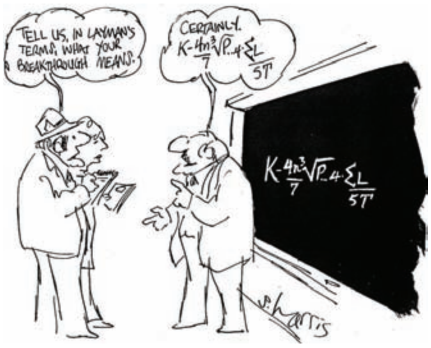
UNIT 9

Expanding algebra

Introduction

So far in the module you have used algebra in a variety of different situations. For example, in Units 6 and 7 you saw how linear models can be used to describe different practical situations, and in Unit 8 you saw how Pythagoras' Theorem together with algebraic manipulations helps to solve various geometric problems. However, many problems in science, technology and other fields lead to more complicated algebra, involving *quadratic expressions*, *algebraic fractions* and *powers*.



In this unit, you will develop further algebraic skills so that you can rearrange more complicated formulas and solve a greater range of equations. In particular, you will learn how to use a formula to obtain the sum of the terms of any *arithmetic sequence* such as

$$3, 7, 11, \ldots, 35,$$

and how to solve *quadratic equations* such as

$$3x^2 + 7x + 4 = 0,$$

by the method of factorisation. You will meet other ways to solve quadratic equations in Unit 10 and also see there how they can be used to solve practical problems involving objects in motion.

This unit contains a large number of examples and activities on basic algebraic techniques. These have been included to give you the opportunity to develop skills that are important for studying further mathematics modules, so try to work through as many as you can.

There are also many tutorial clips on algebraic techniques, and you are recommended to view them to help you understand the techniques.

1 Number patterns and algebra

1.1 Arithmetic sequences

There is a story that the mathematician Carl Friedrich Gauss, when he was 10 years old, surprised his teacher by adding up the natural numbers from 1 to 100 very quickly, giving the answer 5050. The young Gauss is said to have noticed the following efficient way to do this addition.

Gauss is considered to be one of the greatest mathematicians of all time. His method of solving linear equations was mentioned in Unit 7, Section 3.

You can rewrite the sum by rearranging it as a sum of pairs of numbers:

$$1 + 2 + \cdots + 99 + 100 = (1 + 100) + (2 + 99) + \cdots + (50 + 51).$$

Remember that '...' means that something has been left out; it is read as 'dot, dot, dot'.

There are 50 pairs of numbers in brackets, and each pair has sum 101, so

$$1 + 2 + \cdots + 99 + 100 = 50 \times 101 = 5050.$$

Activity 1 *Adding up numbers*

Use the method above to add up the natural numbers from 1 to 200.

What happens if you add up odd numbers? In Section 4 of Unit 1 you were asked to add up the odd numbers

$$1, 3, 5, 7, \ldots,$$

in longer and longer groups, as in the following table.

How many odd numbers	Sum
1	$1 = 1$
2	$1 + 3 = 4$
3	$1 + 3 + 5 = 9$
4	$1 + 3 + 5 + 7 = 16$

On the basis of these examples it appears that if you add up successive odd numbers starting with 1, then the answer is always the square of how many odd numbers you add. For example, in the last line of the table, the first four odd numbers are added and their sum is 16, which is 4^2. This observation led to the following neat formula.

The sum of the first n odd numbers is n^2.

Since there is a formula for the sum of the first n odd numbers, it is reasonable to ask if there is a formula for the sum of the first n natural numbers. Let's look at a table of the first few such sums.

Remember that the natural numbers are

$$1, 2, 3, \ldots.$$

How many numbers	Sum
1	$1 = 1$
2	$1 + 2 = 3$
3	$1 + 2 + 3 = 6$
4	$1 + 2 + 3 + 4 = 10$

In this case a pattern for the sums is not so clear. Figure 1 shows a geometric interpretation of these numbers. You can see that for the first four values of n, the sum of the natural numbers from 1 to n can be represented by a triangular arrangement of dots.

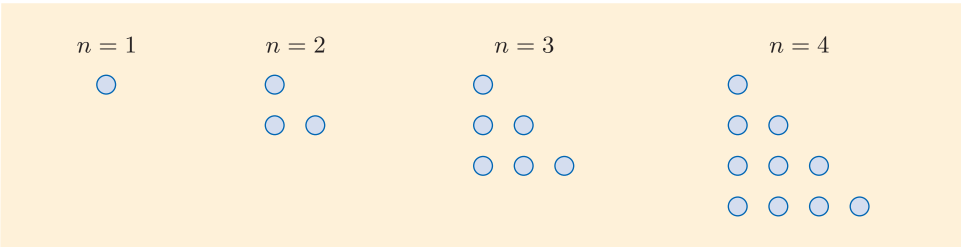


Figure 1 Triangles of dots

It is still not clear from these triangles whether there is a concise formula for the number of dots in the nth triangle. However, by drawing a rotated copy of each of these triangles next to itself, you can obtain rectangles of dots, as shown in Figure 2.

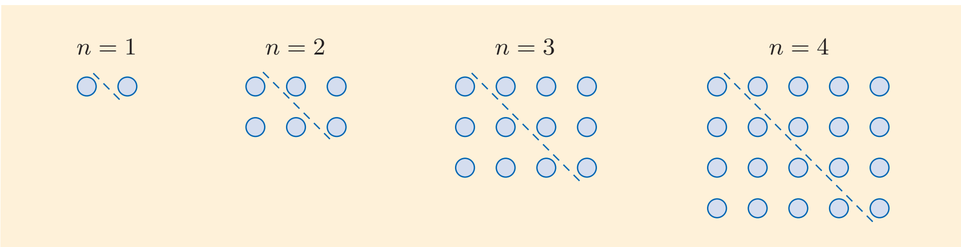


Figure 2 Rectangles of dots

The numbers of dots in these rectangles are

$$1 \times 2 = 2, \quad 2 \times 3 = 6, \quad 3 \times 4 = 12, \quad 4 \times 5 = 20, \quad \ldots .$$

From this construction, you can see that the rectangle constructed from the nth triangle of dots has n rows each with $n+1$ dots, making $n(n+1)$ dots in all. Hence the original triangle must have *half* this many dots, that is, $\frac{1}{2}n(n+1)$ dots. Since the nth triangle has $1 + 2 + \cdots + n$ dots, this gives the formula

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1).$$

The first few values of the expression $\frac{1}{2}n(n+1)$ are shown in the table below.

n	1	2	3	4	5	6	7
$\frac{1}{2}n(n+1)$	1	3	6	10	15	21	28

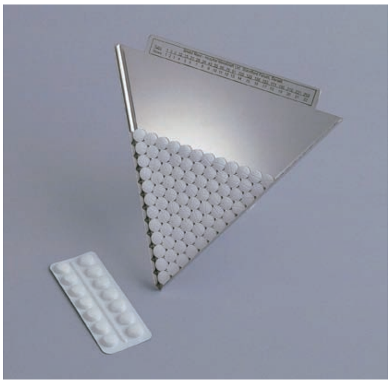


Figure 3 A triangular pill tray

The numbers given by the expression $\frac{1}{2}n(n+1)$ are called **triangular numbers** since they occur as the numbers of dots in the triangular arrays illustrated in Figure 1. These numbers appear in a practical situation as the numbers of pills that can be easily counted in a triangular pill tray. The triangular tray shown in Figure 3 is used because, as pills are put in it, they form a stable triangular pattern. So to count them the pharmacist just needs to check how many rows of pills there are and then read off the corresponding triangular number from a table of these numbers.

You have seen a geometric proof of the formula for the sum of the first n natural numbers. Another way to prove it is to use an algebraic method based on the pairing method of Gauss. As preparation for this proof, here is a slightly different way to write down Gauss' method for adding the first 100 natural numbers.

First, let S denote the sum that you want to find:

$$S = 1 + 2 + \cdots + 99 + 100.$$

Then rearrange the sum in reverse order:

$$S = 100 + 99 + \cdots + 2 + 1.$$

Adding these two equations gives

$$\begin{aligned} 2S = {} & \;\;1 \;+\; 2 \;+ \;\cdots\; + 99 + 100 \\ & + 100 + 99 + \;\cdots\; + \;2\; + \;1. \end{aligned}$$

You can rearrange the expression on the right as a sum of 100 pairs of numbers, giving

$$2S = (1 + 100) + (2 + 99) + \cdots + (99 + 2) + (100 + 1).$$

Each of the 100 pairs in brackets has sum 101. Thus

$$2S = 100 \times 101, \quad \text{so} \quad S = \frac{100 \times 101}{2} = 5050.$$

This reverse order method can be used to find the formula for the sum of the first n natural numbers.

Activity 2 *Finding a formula*

Use the reverse order method to show that

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n + 1).$$

(Hint: Begin by letting S denote the sum of the first n natural numbers.)

Any list of numbers is called a **sequence**. For example, $1, 2, 3, \ldots, 100$ is an example of a **finite sequence**, whereas $1, 2, 3, \ldots$ is an example of an **infinite sequence**. The numbers in a sequence are called the **terms** of the sequence.

An **arithmetic sequence** is a sequence with the property that the **difference** between consecutive terms is constant. For example,

An arithmetic sequence is sometimes called an **arithmetic progression**, and the difference is often called the **common difference**.

$1, 2, 3, \ldots$ is an arithmetic sequence with difference 1,

$10, 8, \ldots, 4, 2, 0$ is an arithmetic sequence with difference -2,

2, 2.5, 3, 3.5, 4 is an arithmetic sequence with difference 0.5.

To specify an arithmetic sequence, we can give

- the first term, denoted by a
- the difference, denoted by d.

If the sequence is finite, we also give the number of terms, denoted by n. The first term a and the difference d can be any number, positive, negative or zero, but the number of terms n is always a positive integer.

For example, looking at each step of the final example given above,

$$2 \xrightarrow{+0.5} 2.5 \xrightarrow{+0.5} 3 \xrightarrow{+0.5} 3.5 \xrightarrow{+0.5} 4,$$

you can see that this sequence can be specified with $a = 2$, $d = 0.5$ and $n = 5$.

Activity 3 *Writing down arithmetic sequences*

(a) Write down the arithmetic sequence with first term 3, difference 7, and 5 terms.

(b) Write down the arithmetic sequence with first term 3, difference -5, and 8 terms.

Let's look at the way that the successive terms of an arithmetic sequence develop. Suppose that the first term is a, the difference is d, and you want to find the nth term. Since the first term is a, the second term is

$$a + d,$$

and the third term is

$$a + d + d = a + 2d,$$

and so on. So the first few terms are

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots.$$

From the first term to the nth term there are exactly $n-1$ additions of d. This gives a formula for the nth term.

The nth term can be thought of as the last term of a finite sequence with n terms.

The nth term of an arithmetic sequence with first term a and difference d is given by the formula

$$n\text{th term} = a + (n-1)d. \qquad (1)$$

For example, for the arithmetic sequence

$$2,\ 2.5,\ 3,\ 3.5,\ \dots,$$

$a = 2$ and $d = 0.5$. So, by equation (1), the 9th term of this sequence is given by

$$2 + (9-1) \times 0.5 = 2 + 8 \times 0.5 = 6,$$

and the 100th term is given by

$$2 + (100-1) \times 0.5 = 2 + 99 \times 0.5 = 51.5.$$

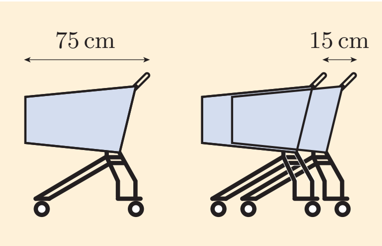


Figure 4 Calculating the length of stacked supermarket trolleys

Activity 4 *Finding the nth term of an arithmetic sequence*

Certain supermarket trolleys are 75 cm long and when stacked each trolley extends 15 cm beyond the one in front, as illustrated in Figure 4.

(a) Write down the lengths (in centimetres) of: one trolley, two stacked trolleys, and three stacked trolleys.

(b) The lengths in part (a) form the first three terms of an arithmetic sequence. Write down its first term a and its difference d.

(c) How long is a line of 20 stacked trolleys? Give your answer in metres.

Suppose that you are given an arithmetic sequence such as

$$3,\ 7,\ 11,\ \dots,\ 35.$$

How can you find the number of terms, n, without writing them all out? If you denote the last term by L, then equation (1) for the nth term can be written as

$$L = a + (n-1)d.$$

Rearranging formulas of this type was covered in Unit 7.

You can find a formula for the number of terms n by rearranging this formula to make n the subject.

Activity 5 *Finding a formula for the number of terms*

Rearrange the formula $L = a + (n-1)d$ to make n the subject. Assume that d is non-zero.

The solution to Activity 5 gives the following formula.

The number of terms n of a finite arithmetic sequence with first term a, last term L and non-zero difference d is given by the formula

$$n = \frac{L - a}{d} + 1. \tag{2}$$

Example 1 *Finding the number of terms*

For the arithmetic sequence

$$3, 7, 11, \ldots, 35,$$

write down the first term a, the difference d and the last term L, and find the number of terms n of the sequence.

Solution

For this sequence, the first term is $a = 3$, the difference is $d = 4$, and the last term is $L = 35$. So, by equation (2), the number of terms is

$$n = \frac{35 - 3}{4} + 1 = 8 + 1 = 9.$$

(Check: The sequence is $3, 7, 11, 15, 19, 23, 27, 31, 35$, which has 9 terms.)

Arithmetic sequences arise in various practical situations. For example, in some jobs the annual salary starts at a particular amount a and increases each year by a fixed amount d up to a certain maximum. Thus if you want to know the *total* amount earned in those years, then you need to find the sum of an arithmetic sequence with first term a and difference d.

This discussion ignores any changes due to annual inflation.

A formula for the sum of any finite arithmetic sequence can be found by using the reverse order method again. The sum of the arithmetic sequence with first term a, difference d and number of terms n is

$$S = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d).$$

By adding this sum to the same sum in reverse order, you obtain

$$\begin{aligned} 2S = a &+ (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) \\ &+ (a + (n - 1)d) + \cdots + (a + 2d) + (a + d) + a. \end{aligned}$$

You can rearrange the expression on the right as a sum of pairs of numbers, each of which has sum $a + a + (n - 1)d$. There are n such pairs, so

$$2S = n(a + a + (n - 1)d) \quad \text{and hence} \quad S = \tfrac{1}{2}n(2a + (n - 1)d).$$

The sum of the finite arithmetic sequence with first term a, difference d and number of terms n is given by the formula

$$S = \tfrac{1}{2}n(2a + (n - 1)d).$$

An alternative version of this formula involving the last term L is

$$S = \tfrac{1}{2}n(a + L).$$

That is, S is n times the mean of the first and last terms.

This formula is often written as

$$S = \frac{n(2a + (n - 1)d)}{2}.$$

You can choose whichever version of this formula is more convenient!

Activity 6 *Summing an arithmetic sequence*

(a) For the arithmetic sequence

$$2, 5, 8, \ldots, 29,$$

write down the first term a, the difference d and the last term L, and hence find the number of terms n of the sequence.

(b) Find the sum of the arithmetic sequence in part (a).

The next activity asks you to use the formula for the sum of an arithmetic sequence in order to compare the total amount of pay offered by two jobs.

Activity 7 *Finding the larger sum*

Two jobs are advertised. In one, the annual salary starts at £20 000 and increases by £500 annually. In the other, the annual salary starts at £18 000 and increases by £1000 annually. Which job pays the greater total amount in the first 10 years?

You should assume that there are no changes due to inflation.

In this section, you've seen formulas for the sums of the first n natural numbers and the first n odd numbers. What about the sum of the first n even numbers:

$$2, 4, \ldots, 2n\,?$$

The numbers $2, 4, \ldots, 2n$ form an arithmetic sequence with first term $a = 2$ and last term $L = 2n$. So you can apply the formula to find the sum S of this sequence, as follows:

$$\begin{aligned} S &= \tfrac{1}{2}n(a+L) \\ &= \tfrac{1}{2}n(2+2n) \\ &= n(n+1). \end{aligned}$$

An alternative way to find the sum of the first n even numbers is to spot that this sum is twice the sum of the first n natural numbers (found in Activity 2), so

$$\begin{aligned} 2+4+\cdots+2n &= 2(1+2+\cdots+n) \\ &= 2\left(\tfrac{1}{2}n(n+1)\right) = n(n+1). \end{aligned}$$

The expression $n(n+1)$ is the number of dots in a rectangle of n rows of $n+1$ dots, as illustrated in Figure 2 on page 73. These numbers are called **oblong numbers** and are twice the triangular numbers, as you can see from Figure 2.

The various types of numbers that you have met have a history going back for thousands of years. Some of them played a role in the theories of the Pythagoreans about the significance of numbers.

The Pythagoreans (about 500 BC) were perhaps the first people to believe that numbers are important in their own right and not just for practical purposes. They named several different types of natural numbers, and linked some of them with properties and superstitions. For example:

- 1 was the source of all numbers
- the odd numbers $1, 3, 5, \ldots$ were masculine and divine
- the even numbers $2, 4, 6, \ldots$ were feminine and earthly
- the prime numbers $2, 3, 5, \ldots$ were one-dimensional (like a line) since a prime number of pebbles cannot be laid out to form a rectangle
- the triangular numbers $1, 3, 6, \ldots$ were considered to be very significant (perhaps because triangles are so important), especially the triangle of 10 dots, called the *tetractys* (Figure 5) which symbolised the four classical elements – earth, air, fire and water.

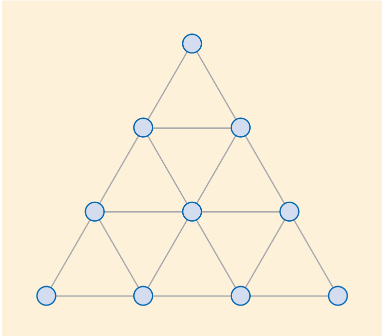


Figure 5 The tetractys

The triangular number 666 has a long history of notoriety. For example, in one translation of the Bible it is referred to as 'the number of the beast'. It is also the sum of the numbers on a roulette wheel, and it appears in various unusual mathematical identities.

These descriptions of numbers seem strange to modern eyes, but the obsession of the Pythagoreans and their later followers with the relationship between numbers and other aspects of life led to many discoveries, such as the role of numbers in determining musical scales.

In more recent times such sequences of numbers, especially prime numbers and triangular numbers, have been the objects of detailed study and found to have many intriguing mathematical properties. Gauss himself was particularly pleased to prove (at the age of 19) that every natural number can be expressed as the sum of at most three triangular numbers.

For example,
$$100 = 1 + 21 + 78.$$

1.2 Another number pattern

Here is another number pattern involving square numbers.

The following table shows values obtained by taking the numbers $1, 2, 3, \ldots$, squaring them, and then subtracting 1.

n	1	2	3	4	5	6	7
n^2	1	4	9	16	25	36	49
$n^2 - 1$	0	3	8	15	24	35	48

Now look at the numbers in the bottom row and consider writing each of them as a product of two factors that are as close together as possible. For example,

$$8 = 2 \times 4, \quad 15 = 3 \times 5 \quad \text{and} \quad 35 = 5 \times 7.$$

In all these examples, the two factors differ by 2 and lie immediately above and below the number that was squared:

$$3^2 - 1 = 8 = 2 \times 4 = (3 - 1)(3 + 1),$$
$$4^2 - 1 = 15 = 3 \times 5 = (4 - 1)(4 + 1),$$
$$6^2 - 1 = 35 = 5 \times 7 = (6 - 1)(6 + 1).$$

You can check that this pattern holds for the other values of n in this table; for example,

$$7^2 - 1 = 48 = 6 \times 8 = (7 - 1)(7 + 1).$$

On the basis of these examples, it is plausible to think that if n is any natural number, then

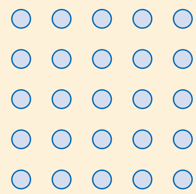
$$n^2 - 1 = (n-1)(n+1).$$

You can check quite quickly that this equation is true for all natural numbers less than 10, say, but does it hold for *all* natural numbers? In other words, is this equation an identity? In the next activity, you are asked to prove that $n^2 - 1 = (n-1)(n+1)$ by a geometric method, similar to one used for some of the other identities that you have met.

Recall from Unit 5 that an *identity* is an equation that is true for all values of the variables.

Activity 8 *Proving a number pattern*

(a) Look at the 5×5 square below, consisting of 25 regularly-spaced dots. By removing the dot in the top right corner and then moving the remaining 4 dots in the top row, construct a rectangle consisting of 24 dots.



Explain how this construction is related to the equation $5^2 - 1 = 4 \times 6$.

(b) The construction in part (a) showed that the equation

$$n^2 - 1 = (n-1)(n+1)$$

is true for $n = 5$. Explain how a similar construction can be used to show that this equation is true for all natural numbers n.

The diagram in the solution to Activity 8 enables you to 'see' why the identity $(n-1)(n+1) = n^2 - 1$ is true. Once you understand why a particular mathematical fact is true, it often happens that you can use similar reasoning to deduce other mathematical facts. For example, you can use similar geometric reasoning with an $n \times n$ square of dots to show that

$$n^2 - 4 = (n-2)(n+2).$$

Try drawing a 5×5 square of dots, removing a 2×2 square of dots from the top right corner, and rearranging the dots to make a rectangle.

However, identities like this one and the one in Activity 8 can also be proved *algebraically* by using the key algebraic skill of 'multiplying out pairs of brackets', as you will see in the next section.

2 Multiplying out pairs of brackets

2.1 Pairs of brackets

In Unit 5 you learned how to multiply out, or expand, brackets in expressions such as

$$2x(-3y + 2z).$$

There, you used the following strategy.

Strategy *To multiply out brackets*

Multiply each term inside the brackets by the multiplier of the brackets.

In the above expression, the multiplier is $2x$.

As a reminder, here are some examples of multiplying out brackets:

$$a(b + c) = ab + ac,$$
$$-2(x - y) = -2x + 2y,$$
$$3m(-2n + 3r - 6) = -6mn + 9mr - 18m.$$

You should also remember that if the multiplier *follows* the brackets, then you can apply the same strategy. The next activity gives you a chance to revise multiplying out brackets.

For example,

$$(b + c)a = ba + ca.$$

Activity 9 *Multiplying out brackets*

Multiply out the brackets in each of the following expressions.

(a) $a(2b + 3c)$ (b) $-r(2s - 3t)$ (c) $(n - 1)n$

The key thing to remember is that you must multiply *each* term in the brackets by the multiplier.

In Section 1 you met an identity that involves a product of two brackets:

$$(n - 1)(n + 1) = n^2 - 1.$$

This identity was proved in Activity 8 by using a geometric argument. But it can also be proved by using algebra to multiply out the brackets in $(n - 1)(n + 1)$. Such products of brackets occur in many situations in mathematics, so it is important to be able to multiply them out correctly.

You can multiply out two brackets of the form

$$(a + b)(c + d)$$

in two steps, as follows.

- First, keep $(a + b)$ as one expression and use it as the multiplier to expand the right bracket $(c + d)$, to obtain

$$(a + b)(c + d) = (a + b)c + (a + b)d.$$

- Second, expand each of the $(a + b)$ brackets on the right-hand side:

$$\begin{aligned}(a + b)(c + d) &= (a + b)c + (a + b)d \\ &= ac + bc + ad + bd.\end{aligned}$$

Alternatively, you could first expand the left bracket $(a + b)$, using $(c + d)$ as the multiplier. You can check that the answer is the same!

If you examine the effect of multiplying out the brackets in $(a+b)(c+d)$, then you can see that each term in the second bracket is multiplied by each term in the first bracket. This always happens when you multiply out two brackets, and it gives the following strategy.

Strategy *To multiply out two brackets*

Multiply each term inside the first bracket by each term inside the second bracket, and add the resulting terms.

When you use this strategy to multiply out

$$(a+b)(c+d),$$

you have to multiply each of the terms a and b in the first bracket by each of the terms c and d in the second bracket. It is a good idea to be systematic about the order in which you do these four multiplications. Figure 6 shows the order that is usually used in this module.

The acronym FOIL may help you to remember the order in which these pairs are multiplied. It stands for:

(1) First: ac,

(2) Outer: ad,

(3) Inner: bc,

(4) Last: bd.

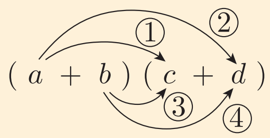


Figure 6 Order of multiplications

Here is the result:

$$(a+b)(c+d) = ac + ad + bc + bd.$$

Note that the answer obtained before the strategy above was a different arrangement of these four terms.

Of course, it is not essential to follow the order given in the above diagram. If you already have some experience of multiplying out pairs of brackets and have developed your own method of doing it, then that's fine as long as you obtain the correct answer!

If a, b, c and d represent positive numbers, then the rule for multiplying out brackets can be thought of in terms of areas of rectangles as follows. In Figure 7 the sides of the large rectangle are $a+b$ units and $c+d$ units, and this rectangle can be split into four smaller rectangles with areas ac, ad, bc and bd square units. Adding these four areas shows that the above expansion of the brackets is correct in this context.

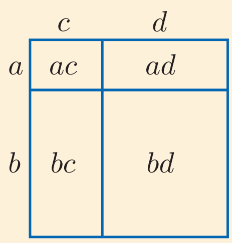


Figure 7 $(a+b)(c+d) = ac + ad + bc + bd$

As usual, you have to be careful when multiplying out brackets if minus signs are present. For example, you can use the strategy above to multiply out the product

$$(2s-t)(u-3v),$$

but you must remember that the terms in the first bracket are $2s$ and $-t$, and the terms in the second bracket are u and $-3v$. It may help to mark these terms, as you did in Section 3 of Unit 5, before multiplying each term in the first bracket by each term in the second bracket:

$$(2s \;\; - t)(u \;\; - 3v) = 2su - 6sv - tu + 3tv.$$

$2s \times u = 2su$,
$2s \times (-3v) = -6sv$,
$(-t) \times u = -tu$,
$(-t) \times (-3v) = 3tv$.

The resulting four terms on the right-hand side have each been simplified in one step, as you learned to do in Unit 5.

Here are some more examples of multiplying out brackets.

Example 2 *Multiplying out two brackets*

Tutorial clip

Multiply out the brackets in each of the following expressions.

(a) $(x+1)(x+2)$ (b) $(x-1)(x-2)$ (c) $(x+1)^2$

(d) $(a+2b)(3c-d)$ (e) $(n-1)(n+1)$

Solution

(a) $(x+1)(x+2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$

(b) $(x-1)(x-2) = x^2 - 2x - x + 2 = x^2 - 3x + 2$

(c) $(x+1)^2 = (x+1)(x+1) = x^2 + x + x + 1 = x^2 + 2x + 1$

(d) $(a+2b)(3c-d) = 3ac - ad + 6bc - 2bd$

(e) $(n-1)(n+1) = n^2 + n - n - 1 = n^2 - 1$

Part (e) of Example 2 is the number pattern identity mentioned in Subsection 1.2, so we have now succeeded in proving this identity algebraically!

The next activity gives you lots of practice in multiplying out brackets.

Activity 10 *Multiplying out two brackets*

Multiply out the brackets in each of the following expressions, simplifying the answer where appropriate.

(a) $(x+2)(x+4)$ (b) $(a+2b)(3x+4y)$ (c) $(x-3)^2$

(d) $(a-2b)(3c-d)$ (e) $(p-1)(-2+3q)$ (f) $(n-2)(n+2)$

The strategy on page 82 can also be used to multiply out two brackets that contain more than two terms. Here a little more care is needed to ensure that all possible pairs of terms are included. For example, in the product

$$(a+b)(c+d+e),$$

each of the two terms in the first bracket must multiply each of the three terms in the second bracket, so there are $2 \times 3 = 6$ product terms in the answer. We obtain these terms in order, beginning with the products involving a and then the products involving b:

$$\begin{aligned}(a+b)(c+d+e) &= a(c+d+e) + b(c+d+e)\\ &= ac + ad + ae + bc + bd + be.\end{aligned}$$

Activity 11 *Multiplying out longer brackets*

Multiply out the brackets in the expression $(2a - b)(c - 3d + 2e)$.

2.2 Squaring brackets

You have now multiplied out many pairs of brackets, including ones involving squaring, such as

$$(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = x^2 + 2x + 1.$$

You often need to square brackets in this way, so it is a good idea to become familiar with this operation. Once again, there is a useful pattern to spot, which you can see in these examples:

$$\begin{aligned}(x + 1)^2 &= x^2 + 2x + 1,\\ (x + 2)^2 &= x^2 + 4x + 4,\\ (x + 3)^2 &= x^2 + 6x + 9,\\ (x + 4)^2 &= x^2 + 8x + 16.\end{aligned}$$

In each case, in the expansion on the right,

- the coefficient of x is *twice* the number in the bracket on the left
- the constant term is the *square* of the number in the bracket on the left.

In general,

$$\begin{aligned}(x + p)^2 &= (x + p)(x + p)\\ &= x^2 + xp + px + p^2\\ &= x^2 + 2px + p^2.\end{aligned}$$

In other words, to square a bracket with two terms, you add together the square of the first term, the square of the last term, and twice the product of the two terms.

A similar pattern occurs when there is a negative sign in the brackets; for example:

$$\begin{aligned}(x - 1)^2 &= x^2 - 2x + 1,\\ (x - 2)^2 &= x^2 - 4x + 4,\\ (x - 3)^2 &= x^2 - 6x + 9,\\ (x - 4)^2 &= x^2 - 8x + 16.\end{aligned}$$

Here the only difference is that a minus sign appears on each side.

Squaring brackets

$$(x + p)^2 = x^2 + 2px + p^2$$

and

$$(x - p)^2 = x^2 - 2px + p^2.$$

So, if you are asked to square brackets, then you may be able to write down the answer directly using one of these general identities (if the expression in brackets is $x + p$ or $x - p$, or of this form). Alternatively, you can multiply

out the brackets in the usual way. Here are some questions for you to try.

Activity 12 *Squaring brackets*

Multiply out the following brackets.

(a) $(x+7)^2$ (b) $(u-10)^2$ (c) $(t+\frac{1}{2})^2$

(d) $(3x+2)^2$ (e) $(2s-5t)^2$

Squaring brackets can also be used to find tricks for squaring numbers in your head. These tricks can even form the basis of a career!

Dr Arthur T. Benjamin is an American professor of mathematics and a professional magician. In a performance called 'Mathemagic', he does rapid mental calculations including squaring 2-, 3- and 4-digit numbers faster than his audience can square them with a calculator. He then explains how he uses simple algebraic techniques to do these mental feats.

Figure 8 Dr Arthur T. Benjamin

To illustrate how algebra can explain such tricks, imagine that you want to square a two-digit number ending in 5. Such a number is of the form $10m+5$, where m is a natural number; for example, $65 = 10\times 6+5$, so $m=6$ in this case, and $45 = 10\times 4+5$, so $m=4$ in this case.

Here is a quick way to calculate the square of a number of this form.

To square $10m+5$, where m is a natural number:

- calculate $m(m+1)$
- the required square is $100m(m+1)+25$.

For example, to find 65^2, where $m=6$:

- $6\times 7 = 42$
- $4200+25 = 4225$.

In the second step you can obtain the answer quickly by putting the digits 25 immediately after the number you found in the first step.

Multiplying a natural number by 100 is the same as writing two zeros at the end of it.

The secret of the trick is just that the square of $10m+5$ is

$$(10m+5)^2 = 100m^2+100m+25.$$

Now $100m^2+100m$ can be factorised as $100m(m+1)$, so the right-hand side can be rearranged to give

$$(10m+5)^2 = 100m(m+1)+25.$$

The key idea here is to replace one apparently complicated calculation by several much easier ones. The difficulty is spotting what those easier calculations are. However, by using algebra you can manipulate one expression into another relatively easily and so not only find a simpler way of doing the calculation but also show that the new calculation works for all numbers of this form. This illustrates some of the power of using algebra and you'll see another example of this in the next section.

There are similar tricks based on algebra for squaring other two-digit numbers. You can find out more about these on the module website.

2.3 Differences of two squares

In Example 2 and Activity 10 on page 83 you were asked to multiply out pairs of brackets in which one bracket contains a sum of two numbers and the other bracket contains a difference of the same two numbers, such as

Here, the sum is $n+1$ and the difference is $n-1$.

$$(n-1)(n+1) = n^2 - 1.$$

This is another type of product that occurs so often in mathematics that it is worth learning the pattern in the answer. Here is the general case:

$$(x-p)(x+p) = x^2 + xp - px - p^2 = x^2 - p^2.$$

Because the form of the right-hand side in this identity is a difference of two squares, namely x^2 minus p^2, the identity is given this name.

Difference of two squares

$$(x-p)(x+p) = x^2 - p^2$$

So, if you are asked to find the product of two brackets of the form above, one a sum and the other the corresponding difference, then you may be able to write down the answer directly using this identity. For example,

Here you replace x by $2n$ and p by 1.

$$(2n-1)(2n+1) = (2n)^2 - 1^2 = 4n^2 - 1.$$

Alternatively, you can just multiply out the brackets in the usual way.

Here are some similar questions for you to try.

Activity 13 *Using a difference of two squares*

Use the difference of two squares identity to multiply out the following brackets.

(a) $(u-12)(u+12)$ (b) $(x-2y)(x+2y)$ (c) $(10-a)(10+a)$

Finally, the following story shows that the difference of two squares identity may help you to win a quiz!

A television quiz once asked a contestant to calculate $51^2 - 49^2$. At first sight, this appears to be quite a tricky calculation:

$$51^2 - 49^2 = 2601 - 2401 = 200.$$

However, by using a difference of two squares, the answer can be found in your head:

$$51^2 - 49^2 = (51-49)(51+49) = 2 \times 100 = 200.$$

3 Quadratic expressions and equations

Earlier in the module you learned how to *solve* various types of equations, that is, how to find the values of the unknowns in them. For example, you saw in Unit 5 how to solve equations such as

$$3(x+1) = 12,$$

which is a single linear equation in the unknown x, and in Unit 7 how to solve

$$3x + 2y = 11,$$
$$2x - 3y = 3,$$

which is a pair of simultaneous linear equations in the unknowns x and y.

In this section you will meet a new type of equation, called a *quadratic equation*, and learn one way that such an equation can often be solved.

3.1 Quadratic expressions

If you expand the brackets in the product $(x+1)(3x-4)$, then you obtain

$$\begin{aligned}(x+1)(3x-4) &= 3x^2 - 4x + 3x - 4\\ &= 3x^2 - x - 4.\end{aligned}$$

The resulting expression involves three terms:

- a term in x^2, namely $3x^2$
- a term in x, namely $-x$
- a constant term, namely -4.

An expression of the form

$$ax^2 + bx + c, \quad \text{where } a, b, c \text{ are numbers, and } a \neq 0,$$

is called a **quadratic expression** in x, or a quadratic in x, or just a **quadratic**.

The numbers a, b and c are called the **coefficients** of the quadratic.

Remember that the statement $a \neq 0$ is read as 'a is not equal to 0' or as 'a is non-zero'.

The Latin word 'quadrare' means 'to square'.

The coefficients b and c in a quadratic expression $ax^2 + bx + c$ can equal 0, but a must be non-zero, so that the expression includes a term in x^2. Also, you can write the terms of a quadratic in any order, but in this module we usually put them in the order above, with the term in x^2 first.

As usual, the variable in a quadratic expression can be any letter. For example:

- $x^2 + 2x - 3$ is a quadratic in x, with $a = 1$, $b = 2$ and $c = -3$
- $3t^2 + 1$ is a quadratic in t, with $a = 3$, $b = 0$ and $c = 1$
- $2x + 1$ is not a quadratic, as there is no term in x^2
- $3y^3 + y^2 + 1$ is not a quadratic, because it includes a power of y higher than a square.

Activity 14 *Identifying quadratic expressions*

Which of the following expressions are quadratics? For those that are quadratics, state the values of a, b and c.

(a) $9x^2 - 12x + 4$ (b) $3y - 5$ (c) $-6 - 7s^2$ (d) $2x^3 + x^2$

3.2 Quadratic equations

Any equation that can be expressed in the form

$$ax^2 + bx + c = 0$$

(by rearranging if necessary) is called a **quadratic equation** in x. In this equation, x is an unknown, and a, b and c are numbers with $a \neq 0$. One of the key techniques of algebra that you will learn in this module is how to solve a quadratic equation, that is, how to find the values of x that satisfy the equation.

Activity 15 *Checking a solution*

Show that $x = 2$ is a solution of the quadratic equation

$$2x^2 - x - 6 = 0.$$

There are many problems in mathematics that require you to solve a quadratic equation. You have already met some examples in Section 3 of Unit 8, where you used Pythagoras' Theorem to find side lengths in right-angled triangles. You will meet several more examples in this unit and the next.

The importance of quadratic equations has sometimes been called into question. For example, there was a debate in Parliament in 2003 about quadratic equations and whether they were 'irrelevant'. The closing speech of the debate included the following stirring words in support of the teaching of quadratic equations, and of mathematics in general!

> 'Quadratic equations allow us to analyse the relationships between variable quantities, and they are the tool for understanding variable rates of change. It is in variable rates of change that quadratic equations are seen in economics, science and engineering. Examples of the use of quadratic equations include acceleration, ballistics and financial comparisons ...
>
> In conclusion, the teaching of quadratic equations, and of the mathematics curriculum overall, is key to a future work force that can develop and use mathematical models in daily life. As research in a book of quotations reveals, Napoleon said: "The advancement and perfection of mathematics are intimately connected with the prosperity of the state."'
>
> Alan Johnson, Minister for Lifelong Learning, Further and Higher Education, 26 June 2003.

So how do quadratic equations occur? In Units 5 and 7 you saw that equations arise from problems in which you have to find an unknown number, by using the following strategy.

Strategy *To find an unknown number*

- Represent the number that you want to find by a letter.
- Express the information that you know about the number as an equation.
- Solve the equation.

Here is a problem (not a practical problem, however) in which finding an unknown number leads to a quadratic equation.

> Find a number with the property that if you square the number and add 6, then the answer is 5 times the number.

If you try some simple numbers, then you soon find that one answer is 2, because

$$2^2 + 6 = 10 \quad \text{and} \quad 5 \times 2 = 10.$$

But if you continue to search, then you find another solution, namely 3, because

$$3^2 + 6 = 15 \quad \text{and} \quad 5 \times 3 = 15.$$

The fact that there is more than one solution to this problem may seem strange at first – could there be further solutions?

To investigate, we can follow the strategy above. First, let x represent a number that has the given property. Then $x^2 + 6$ must be the same as $5x$; that is,

$$x^2 + 6 = 5x.$$

This equation can be rearranged to give the quadratic equation

$$x^2 - 5x + 6 = 0.$$

Thus if you know how to solve a quadratic equation, then you can solve the problem. In this case you can find two solutions without using the quadratic equation, as you saw, but if the numbers had been different, then using the quadratic equation might have been essential.

The above problem is just a puzzle, but it is intriguing because there are two possible solutions. In fact, it illustrates a type of problem used by Babylonian teachers almost 4000 years ago as part of the mathematical training of their scribes! For example, one of their problems was:

> If the sum of the area of a square and its side length is $\frac{3}{4}$, then what is the side length of the square?

When using mathematics to solve a *practical* problem, you should not add an area to a length, as the units are different.

The ancient Babylonians developed many skills such as agriculture, irrigation, writing and arithmetic. Their systems of trade required reliable methods of calculating areas and volumes, to find the volumes of grain in storage containers, for example. So they invented many techniques for solving numerical problems, including ones leading to quadratic equations. The problems that they used for teaching were often puzzles. Other problems involved measurements of fields and construction projects, usually in somewhat unrealistic settings.

You will see more examples of how quadratic equations arise later in this section, after you have met a method of solving them.

3.3 Solving simple quadratic equations

Some quadratic equations can be solved quite easily. For example, the simple equation

$$x^2 = 0$$

has the solution $x = 0$, and this is the only solution of this equation. The equation

$$x^2 - 4 = 0, \quad \text{or equivalently} \quad x^2 = 4,$$

can also be solved easily. It has the solution $x = 2$, since $2^2 = 4$, and it also has the solution $x = -2$, since $(-2)^2 = 4$.

You can write both of these solutions together as $x = \pm 2$.

The equation

$$x^2 = 2$$

Square roots were discussed in Unit 3.

can be solved in a similar way. It has the solution $x = \sqrt{2}$, since $(\sqrt{2})^2 = 2$, and it also has the solution $x = -\sqrt{2}$, since $(-\sqrt{2})^2 = 2$.

More generally, every equation of the form

$$x^2 = d, \quad \text{where } d > 0,$$

has two solutions, namely $x = \pm\sqrt{d}$.

Tutorial clip

Example 3 *Solving simple quadratic equations*

Solve each of the following quadratic equations.

(a) $x^2 - 9 = 0$ (b) $t^2 - 10 = 0$

Solution

(a) This equation can be rearranged as $x^2 = 9$, so it has two solutions, $x = \pm 3$.

(b) This equation can be rearranged as $t^2 = 10$, so it has two solutions, $t = \pm\sqrt{10}$.

Here are two quadratic equations of this type for you to solve.

Activity 16 *Solving simple quadratics*

Solve each of the following quadratic equations.

(a) $x^2 - 25 = 0$ (b) $t^2 - 3 = 0$

Simple quadratic equations of this form are sometimes used by the police when investigating road traffic accidents. By measuring the length of a vehicle's skid mark, the speed of the vehicle at the start of the skid can be estimated by using the formula $s^2 = c\,d$, where d is the length of the skid mark in metres, s is the speed at the start of the skid in km/h, and c is a

constant that depends on factors like the road surface and the condition of the vehicle. So the speed s is equal to $\sqrt{cd}$.

In contrast to these simple quadratic equations, there are others that seem to be impossible to solve. For example, the equation

$$x^2 + 1 = 0, \quad \text{or equivalently} \quad x^2 = -1,$$

has no solutions, at least among the real numbers, because when you square any real number the answer is not negative.

In higher-level mathematics modules, so-called *complex numbers* are introduced in order to provide solutions to equations like $x^2 + 1 = 0$.

So, even among the simple quadratic equations in this subsection, there are equations that have one solution, two solutions or no solutions! These three different possibilities occur also for more complicated quadratic equations. You will see in Unit 10 why a quadratic equation never has more than two solutions. All the quadratic equations that you are asked to solve in this module have at least one solution.

3.4 Factorising quadratics of the form $x^2 + bx + c$

In this subsection you will learn a technique that can often be used to solve a quadratic equation. The technique is to **factorise** the quadratic expression that appears in the equation; that is, to express the quadratic expression as a product of simpler expressions. Initially, we consider only quadratic expressions of the form $x^2 + bx + c$ (so the coefficient of x^2 is 1), where b and c are *integers*.

You saw earlier that multiplying out brackets like $(x + 2)(x + 3)$ leads to a quadratic expression:

$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6.\end{aligned}$$

This shows that $x^2 + 5x + 6$ can be written as a product of the simpler expressions $x + 2$ and $x + 3$, each of which is called a **factor** of $x^2 + 5x + 6$.

But now suppose that you want to do the reverse process; that is, you want to *find* a factorisation of $x^2 + 5x + 6$:

$$x^2 + 5x + 6 = (\qquad)(\qquad),$$

where each of the two expressions in brackets on the right-hand side contains a term in x and a constant term. Because the term in x^2 in the quadratic expression is just x^2, the factorisation must be of the form

$$x^2 + 5x + 6 = (x \sqcup\!\!\sqcup)(x \sqcup\!\!\sqcup),$$

where there is a positive or negative constant term in each of the gaps indicated. You can find these missing numbers by comparing the coefficients of the terms on both sides of the equation. When you multiply out the brackets on the right-hand side:

- the constant term that you get is the product of the two missing numbers, so this product is equal to the constant term on the left-hand side, which is 6
- the coefficient of the term in x that you get is the sum of the two missing numbers, so this sum is equal to the coefficient of x on the left-hand side, which is 5.

You can see these two facts happening in practice if you look at the working for $(x + 2)(x + 3)$ above, or try some similar examples yourself.

It is not hard to guess a pair of numbers whose product is 6 and whose sum is 5, namely $2, 3$, so this gives the required factorisation:

$$x^2 + 5x + 6 = (x + 2)(x + 3),$$

as expected.

See Unit 3 for a discussion of factor pairs.

A more systematic approach is to consider all factor pairs of 6 and choose the pair whose sum is 5. The only factorisations of 6 as a product of two positive numbers are

$$1 \times 6 \quad \text{and} \quad 2 \times 3.$$

Among these factor pairs, only the pair $2, 3$ has sum 5.

This approach is the basis of the following strategy for trying to factorise any quadratic expression of the form $x^2 + bx + c$, where b and c are integers.

Strategy *To factorise $x^2 + bx + c$, where b and c are integers*

Fill in the gaps in the brackets on the right-hand side of the equation

$$x^2 + bx + c = (x \underline{\quad})(x \underline{\quad})$$

with two numbers whose product is c and whose sum is b:

$$x^2 + \underset{\substack{\uparrow \\ \text{sum}}}{b}x + \underset{\substack{\uparrow \\ \text{product}}}{c}.$$

You can search systematically for integers with these properties as follows:

- write down the factor pairs of c, the constant term
- choose (if possible) a pair whose sum is b, the coefficient of x.

If there is no pair of integers with these two properties, then a factorisation of the quadratic of the form $x^2 + bx + c = (x \underline{\quad})(x \underline{\quad})$, with integers in the gaps, is not possible. In this unit we concentrate on those quadratics that *can* be factorised using integers, but you should be aware that this is not possible for all quadratics.

Here is an example of this strategy in action.

Tutorial clip

Example 4 *Factorising a quadratic expression*

Factorise the quadratic expression $x^2 + 6x + 8$.

Solution

Find a pair of numbers whose product is 8 and whose sum is 6.

The positive factor pairs of 8 are

$$1, 8, \quad 2, 4.$$

The only pair whose sum is 6 is $2, 4$.

If you spot that the pair $2, 4$ has product 8 and sum 6 straight away, then there is no need to write down any other factor pairs.

Thus

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x + 2)(x + 4) &= x^2 + 4x + 2x + 8 \\ &= x^2 + 6x + 8.)\end{aligned}$$

When using the strategy on the opposite page, you will often need to consider factor pairs of the constant term c that include negative factors; for example, $-1, -8$ and $-2, -4$ are factor pairs of 8. However, in Example 4 you needed to consider only positive factor pairs, because:

- the product of the factors, 8, is positive, so the factors must have the same sign as each other
- the sum of the factors, 6, is positive, so *both factors must be positive.*

So you needed to consider only the factor pairs $1, 8$ and $2, 4$.

The quadratics for you to factorise in Activity 17 are of this type.

The strategy on the opposite page suggests that you write down the possible factor pairs systematically. However, if you can spot the required factor pair, then there is no need to write down the others.

Activity 17 *Factorising quadratic expressions*

Factorise each of the following quadratic expressions.

(a) $x^2 + 3x + 2$ (b) $x^2 + 11x + 24$

Don't forget to check that your factorisations work!

In the solution to Activity 17(b), all positive factor pairs of 24 were listed for completeness. But when you solve such a problem, you could omit the pair $1, 24$, for example, since its sum is clearly not 11. Your aim is to find a factor pair that does work!

In the next example, the constant term c is again positive but b, the coefficient of x, is negative. In this case, you need to consider only negative factor pairs, because:

- the product of the factors is positive, so the factors must have the same sign as each other
- the sum of the factors is negative, so *both factors must be negative.*

Example 5 *Factorising a quadratic expression*

Tutorial clip

Factorise the quadratic expression $x^2 - 5x + 6$.

Solution

Find a pair of numbers whose product is 6 and whose sum is -5.

The negative factor pairs of 6 are

$$-1, -6, \quad -2, -3.$$

The only pair whose sum is -5 is $-2, -3$.

Thus

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x - 2)(x - 3) &= x^2 - 3x - 2x + 6\\ &= x^2 - 5x + 6.)\end{aligned}$$

Here are some quadratics of this type for you to factorise.

Activity 18 *Factorising quadratic expressions*

Factorise each of the following quadratic expressions.

(a) $x^2 - 10x + 24$ (b) $t^2 - 4t + 3$ (c) $x^2 - 6x + 9$

In the next example, the constant term c is negative, so the numbers in the factor pairs of c must have opposite signs. This leads to more factor pairs than if c is positive.

Tutorial clip

Example 6 *Factorising a quadratic expression*

Factorise the quadratic expression $x^2 - 7x - 8$.

Solution

Find a pair of numbers whose product is -8 and whose sum is -7.

The factor pairs of -8 are

$$1, -8, \quad 2, -4, \quad -1, 8, \quad -2, 4.$$

The only pair whose sum is -7 is $1, -8$.

Thus

$$x^2 - 7x - 8 = (x + 1)(x - 8).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x + 1)(x - 8) &= x^2 - 8x + x - 8\\ &= x^2 - 7x - 8.)\end{aligned}$$

Here are some quadratics of this type for you to factorise.

Activity 19 *Factorising quadratic expressions*

Factorise each of the following quadratic expressions.

(a) $x^2 - x - 2$ (b) $u^2 + 4u - 12$

Factorising special quadratic expressions

Some special quadratic expressions can be factorised more easily. For example, if there is no constant term (that is, if $c = 0$), then x is always a common factor. For example:

$$x^2 + 4x = x(x + 4),$$
$$x^2 - 6x = x(x - 6).$$

Another special factorisation occurs when the quadratic expression is a difference of two squares. For example:

$$x^2 - 1 = (x - 1)(x + 1), \quad \text{because} \quad x^2 - 1 = x^2 - 1^2,$$
$$x^2 - 9 = (x - 3)(x + 3), \quad \text{because} \quad x^2 - 9 = x^2 - 3^2.$$

Finally, you can sometimes recognise that a quadratic expression is a **perfect square**; that is, it is equal to the square of a simpler expression, because it is of the form

$$x^2 + 2px + p^2 \quad \text{or} \quad x^2 - 2px + p^2.$$

In these cases

$$x^2 + 2px + p^2 = (x + p)^2 \quad \text{or} \quad x^2 - 2px + p^2 = (x - p)^2,$$

as you saw in Subsection 2.2. For example, the quadratic expression $x^2 - 6x + 9$ from Activity 18 is of the form

$$x^2 - 2px + p^2, \quad \text{with } p = 3.$$

This shows, without using the strategy, that

$$x^2 - 6x + 9 = (x - 3)^2.$$

Activity 20 *Factorising special quadratic expressions*

Factorise each of the following quadratic expressions.

(a) $x^2 - x$ (b) $u^2 - 16$ (c) $t^2 - 9t$ (d) $x^2 + 10x + 25$

3.5 Solving quadratic equations by factorisation

Suppose now that you want to solve the quadratic equation

$$x^2 - 5x + 6 = 0. \tag{3}$$

In Example 5 you saw that you can factorise the quadratic expression

$$x^2 - 5x + 6 \quad \text{as} \quad (x - 2)(x - 3).$$

So you can rewrite equation (3) as

$$(x - 2)(x - 3) = 0.$$

How does this rewriting of the equation help? Well, this new equation states that the product of the two numbers $x - 2$ and $x - 3$ is 0. So you can use the following property of numbers.

If the product of two or more numbers is 0, then at least one of the numbers must be 0.

This property is true because if two numbers are both non-zero, then their product is non-zero.

You can apply this property to the above factorisation to deduce that

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0.$$

If $x - 2 = 0$, then $x = 2$, and if $x - 3 = 0$, then $x = 3$. So the quadratic equation (3) has two solutions:

$$x = 2 \quad \text{and} \quad x = 3.$$

Here is a check that these values of x do indeed satisfy equation (3):

$$\text{when } x = 2, \quad x^2 - 5x + 6 = 2^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0,$$
$$\text{when } x = 3, \quad x^2 - 5x + 6 = 3^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0.$$

In general, you can use the following strategy.

Strategy *To solve $x^2 + bx + c = 0$ by factorisation*

1. Find a factorisation:
$$x^2 + bx + c = (x + p)(x + q).$$
2. Then $(x + p)(x + q) = 0$, so
$$x + p = 0 \quad \text{or} \quad x + q = 0,$$
and hence the solutions are
$$x = -p \quad \text{and} \quad x = -q.$$

The numbers p and q may be positive, negative or zero.

Note that the two solutions of a quadratic equation may be the same; in this case the equation is said to have a **repeated solution**.

Once you have found the solutions, it is a good idea to check that they both satisfy the equation $x^2 + bx + c = 0$.

Here is an example of this strategy in action.

Tutorial clip

Example 7 *Solving a quadratic equation*

Solve $x^2 - 7x - 8 = 0$ by factorisation.

Solution

The equation is: $x^2 - 7x - 8 = 0$

Factorise: $(x + 1)(x - 8) = 0$

So: $x + 1 = 0$ or $x - 8 = 0$

So: $x = -1$ or $x = 8$

This factorisation was found in Example 6.

(Check: When $x = -1$,
$$x^2 - 7x - 8 = (-1)^2 - 7 \times (-1) - 8 = 1 + 7 - 8 = 0.$$
When $x = 8$,
$$x^2 - 7x - 8 = 8^2 - 7 \times 8 - 8 = 64 - 56 - 8 = 0.)$$

Here are some quadratic equations for you to solve.

Activity 21 *Solving equations by factorisation*

Solve each of the following quadratic equations by factorisation.

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 10x + 24 = 0$ (c) $t^2 - 16 = 0$

(d) $u^2 - u - 12 = 0$ (e) $x^2 - 6x + 9 = 0$ (f) $x^2 - 9x = 0$

You factorised several of these quadratics in earlier activities.

Factorisation is an efficient method of solving quadratic equations when it can be applied. However, as stated earlier, not all quadratics of the form $x^2 + bx + c$ can be factorised in the form $(x + p)(x + q)$ using integers p and q. In Unit 10 you will meet a formula for solving a quadratic equation which avoids the need for factorisation.

3.6 Factorising quadratics of the form $ax^2 + bx + c$

You have now seen how to factorise quadratic expressions of the form $x^2 + bx + c$, whenever this is possible using integers. It's also possible to factorise many expressions in which the coefficient of x^2 is not 1, as you will now see.

Sometimes, factorising a quadratic expression of the form $ax^2 + bx + c$, where a is not 1, can be reduced to the case when the first term is x^2 because a is a common factor of the coefficients. For example, in the quadratic $2x^2 + 10x + 12$, each of the coefficients is a multiple of 2, so

$$2x^2 + 10x + 12 = 2(x^2 + 5x + 6).$$

You saw earlier that $x^2 + 5x + 6 = (x + 2)(x + 3)$, so

$$2x^2 + 10x + 12 = 2(x + 2)(x + 3).$$

Activity 22 *Factorising when the coefficients have a common factor*

Factorise each of the following quadratic expressions.

(a) $3x^2 - 3x - 36$ (b) $-5x^2 + 15x - 10$

However, consider the quadratic expression

$$2x^2 - x - 6.$$

The presence of the coefficient 2 in the term $2x^2$ means that a factorisation of the form $(x + p)(x + q)$ is not possible; also, 2 is not a common factor of the coefficients. A factorisation using integers *may* still be possible though, and below are two methods that you can use to try to find one. You can use either method, but if you have already met one of them and are confident in using it, then you may prefer to continue to use your method.

Remember that many quadratics do not factorise using integers.

The first method is based on checking all possibilities.

Example 8 *Factorising a general quadratic expression – first method*

Tutorial clip

Factorise $2x^2 - x - 6$.

Solution

First note that the terms in x in the brackets must be $2x$ and x. Then try to find a factorisation of the form

$$2x^2 - x - 6 = (2x \_\_\_)(x \_\_\_),$$

where the gaps each contain a positive or negative integer.

The two missing integers must have product -6.

The possible factor pairs of -6 are

$$-1, 6, \quad 1, -6, \quad -2, 3, \quad 2, -3.$$

These four factor pairs lead to eight possible cases:

$$\begin{aligned}
&(2x - 1)(x + 6) \quad \text{or} \quad (2x + 6)(x - 1),\\
&(2x + 1)(x - 6) \quad \text{or} \quad (2x - 6)(x + 1),\\
&(2x - 2)(x + 3) \quad \text{or} \quad (2x + 3)(x - 2),\\
&(2x + 2)(x - 3) \quad \text{or} \quad (2x - 3)(x + 2).
\end{aligned}$$

By multiplying out each of these pairs of brackets in turn, you find that one of these cases is the required factorisation, specifically,

$$\begin{aligned}(2x+3)(x-2) &= 2x^2-4x+3x-6\\ &= 2x^2-x-6.\end{aligned}$$

Thus

$$2x^2-x-6=(2x+3)(x-2).$$

You can often use this first method efficiently by deciding which are the most likely cases (for example, by considering which signs are possible) and checking these cases first. But the method can be time consuming because there may be many cases to consider. For example, if the first term of the quadratic is $6x^2$, then you have to consider brackets starting with $6x$ and x, and also with $3x$ and $2x$.

In the second method, you need to consider fewer cases but it is not immediately clear why the method works. Indeed, explaining why it works requires a higher level of mathematics than is appropriate in this module. At this stage you should concentrate on learning the technique, which is described in the green text.

Tutorial clip

Example 9 *Factorising a general quadratic expression – second method*

Factorise $2x^2-x-6$.

Solution

The quadratic expression is of the form ax^2+bx+c, where $a=2$, $b=-1$ and $c=-6$.

First, find two numbers whose product is ac and whose sum is b.

For this quadratic expression, $ac=2\times(-6)=-12$ and $b=-1$.

The possible factor pairs of -12 are

$$-1, 12,\quad 1,-12,\quad -2,6,\quad 2,-6,\quad -3,4,\quad 3,-4.$$

The only pair whose sum is -1 is $3,-4$.

Next, rewrite the quadratic expression, splitting the term in x using the above factor pair.

Since $-1=3-4$,

$$2x^2-x-6=2x^2+3x-4x-6.$$

Finally, group the four terms in pairs and take out common factors to give the required factorisation.

Then

$$\begin{aligned}2x^2-x-6 &= \underbrace{2x^2+3x}\ \underbrace{-4x-6}\\ &= x(2x+3)-2(2x+3)\\ &= (x-2)(2x+3).\end{aligned}$$

Here, the first pair of terms on the RHS has a common factor of x and the second pair has a common factor of -2, and then both expressions $x(2x+3)$and $-2(2x+3)$ have a common factor of $(2x+3)$.

Thus

$$2x^2-x-6=(x-2)(2x+3).$$

Note that the second method works in whichever order you split the middle term:

$$\begin{aligned}2x^2 - x - 6 &= 2x^2 - 4x + 3x - 6\\ &= 2x(x-2) + 3(x-2)\\ &= (2x+3)(x-2).\end{aligned}$$

In the following activity you can use either the first method or the second method.

Activity 23 *Factorising a general quadratic expression*

Factorise each of the following quadratic expressions.

(a) $2x^2 - 5x + 3$ (b) $6x^2 + 7x - 3$ (c) $8x^2 - 10x - 3$

Once you have factorised a quadratic expression, you can solve the corresponding quadratic equation. In Examples 8 and 9 you saw that

$$2x^2 - x - 6 = (2x+3)(x-2).$$

Therefore the equation

$$2x^2 - x - 6 = 0$$

can be written as

$$(2x+3)(x-2) = 0.$$

Hence the solutions of this equation satisfy

$$2x + 3 = 0 \quad \text{or} \quad x - 2 = 0.$$

If $2x + 3 = 0$, then $2x = -3$ so $x = -\frac{3}{2}$.

If $x - 2 = 0$, then $x = 2$.

Hence the solutions are

$$x = -\tfrac{3}{2} \quad \text{and} \quad x = 2.$$

Here are some general quadratic equations for you to solve.

Activity 24 *Solving general quadratic equations*

Use your answers to Activity 23 to solve each of the following quadratic equations.

(a) $2x^2 - 5x + 3 = 0$ (b) $6x^2 + 7x - 3 = 0$ (c) $8x^2 - 10x - 3 = 0$

Before you start to solve a quadratic equation it is a good idea to check that it is in its simplest form as this can make it easier to work with. Here are some things that you can do to simplify a quadratic equation.

Simplifying a quadratic equation

- If the coefficient of x^2 is negative, then multiply the equation through by -1 to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.

For example, to solve the equation

$$-5x^2 + 15x - 10 = 0,$$

you can multiply through by -1 to obtain

$$5x^2 - 15x + 10 = 0.$$

Then divide the equation through by the common factor 5 to obtain

$$x^2 - 3x + 2 = 0,$$

which factorises to give

$$(x - 1)(x - 2) = 0.$$

So the solutions of this equation are $x = 1$ and $x = 2$.

If the sum of the area of a square and its side length is $\frac{3}{4}$, then what is the side length of the square?

Finally, the ancient Babylonian problem mentioned on page 89 leads to the equation

$$x^2 + x = \tfrac{3}{4}, \quad \text{which can be rearranged as} \quad x^2 + x - \tfrac{3}{4} = 0.$$

You can obtain an equivalent quadratic equation with coefficients that are integers by clearing fractions in the usual way. Multiplying through by 4 gives the equation

$$4x^2 + 4x - 3 = 0.$$

This quadratic expression can be factorised using either the first or second method to give

$$4x^2 + 4x - 3 = (2x - 1)(2x + 3).$$

The solutions satisfy $2x - 1 = 0$ or $2x + 3 = 0$, so they are $x = \frac{1}{2}$ and $x = -\frac{3}{2}$. However, the problem was to find the side length of a square, so x is positive. Thus the answer to this problem is $x = \frac{1}{2}$.

3.7 Problems leading to quadratic equations

Quadratic equations arise in many problems, as you will now see.

Example 10 *Using a quadratic equation*

Two walkers start from the same point. The first walks east at 4 mph. The second starts half an hour after the first and walks north, also at 4 mph. How long after the first walker sets out are they 10 miles apart?

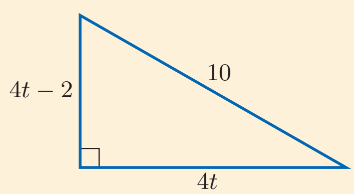
Solution

Let t denote the time in hours after which the walkers are 10 miles apart.

Remember that distance = speed × time.

After t hours the first walker has covered $4t$ miles. The second walker starts half an hour later, so after t hours has walked for $t - \frac{1}{2}$ hours, and so has covered $4(t - \frac{1}{2}) = 4t - 2$ miles.

This diagram shows the situation when the two walkers are 10 miles apart.



By Pythagoras' Theorem,

$$(4t)^2 + (4t-2)^2 = 10^2. \qquad (4)$$

Pythagoras' Theorem was discussed in Unit 8, Section 3.

Now $(4t)^2 = 16t^2$ and $(4t-2)^2 = 16t^2 - 16t + 4$, so equation (4) can be written as

$$16t^2 + 16t^2 - 16t + 4 = 100.$$

This equation can be rearranged as

$$32t^2 - 16t - 96 = 0; \quad \text{that is,} \quad 2t^2 - t - 6 = 0,$$

Here, both sides are divided by 16.

which is a quadratic equation.

The factorisation of $2t^2 - t - 6$ was found in Example 8:

In Example 8 the variable was x rather than t.

$$2t^2 - t - 6 = (2t+3)(t-2).$$

This factorisation shows that the solutions of this equation satisfy

$$2t + 3 = 0 \quad \text{or} \quad t - 2 = 0.$$

Hence $t = -\frac{3}{2}$ or $t = 2$.

Since a negative number makes no sense as a solution here, the answer to the question is 2 hours.

(Check: After 2 hours the first walker has gone 8 miles and the second has gone 6 miles, so, by Pythagoras' Theorem, their distance apart is indeed $\sqrt{8^2 + 6^2} = 10$ miles.)

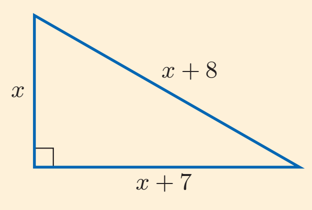
In Example 10, a negative number was rejected as a possible answer. In general, when answering mathematical questions by solving equations you should use the following principle.

If a mathematical question leads to an equation with more than one solution, then you should accept as possible answers only those solutions that make sense for the original question.

Here are some more questions that can be answered by finding a suitable quadratic equation and solving it using factorisation.

Activity 25 *Finding an unknown side length*

The diagram shows a right-angled triangle. The side lengths are measured in centimetres.

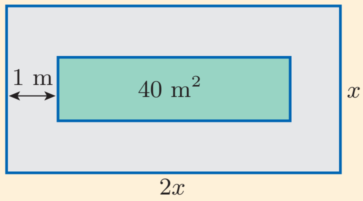


(a) Apply Pythagoras' Theorem to obtain an equation involving x.

(b) Rearrange the equation to obtain a quadratic equation for x, and hence find the value of x.

Activity 26 *Finding the width of a garden*

The diagram shows a rectangular garden that is twice as long as it is wide. Around the edge is a path 1 m wide that surrounds a rectangular lawn with area $40\,\text{m}^2$.



How wide is the garden?

To end this section, here are some details of the life of one of the first people to use quadratic equations, and many other types of equations, to solve real-world problems.

Figure 9 Thomas Harriot (1560–1621)

Thomas Harriot was an English astronomer and mathematician. He worked for Sir Walter Raleigh (also spelt Ralegh), providing mathematical information on practical matters such as navigation and the optimal method of stacking cannonballs, and later for Henry Percy, Duke of Northumberland, who had a great interest in scientific questions. Unfortunately, Harriot published very little, but he is now credited with being the first person to view the Moon through a telescope and make a drawing of it, the first to view sunspots, and the first to state Snell's law of refraction in optics.

Harriot taught courses on navigation (based on 'spherical' trigonometry) to Raleigh's seamen, and in mathematics he established many of the basics of algebra needed to solve quadratic equations and equations involving higher powers of the unknown x, such as x^3, drawing on the work of the French mathematician Viète.

Harriot introduced a simplified notation for doing algebra (though he still wrote a^2 as aa, a^3 as aaa, and so on) and he understood the idea of factorisation of quadratics and expressions involving higher powers of the unknown x. He also used negative solutions of equations and even solutions that involved square roots of negative numbers.

In this way, Harriot anticipated the much later development of complex numbers.

4 Manipulating algebraic fractions

As you saw in Unit 5, fractions that involve algebraic expressions are called **algebraic fractions**. Here are some examples:

$$\frac{8ab}{c}, \quad \frac{1}{(x-3)^2} \quad \text{and} \quad \frac{2x+1}{x^2+1}.$$

Algebraic fractions appear in many important formulas. For example,

$$F = \frac{GmM}{r^2}$$

is a formula for the force of gravitational attraction F between two objects that have masses m and M, and are a distance r apart; here G is a constant, called the *universal gravitational constant*, measured in suitable units.


"I'd like to meet you halfway, but I'm terrible with fractions."

To be able to work with such formulas, you need to be able to manipulate algebraic fractions in a similar way to numerical fractions. For example, you should be able to add them and multiply them. In this section you will practise working with algebraic fractions, and in the next section you will meet some problems that are solved by manipulating algebraic fractions.

When you work with algebraic fractions, there is a risk of considering values of the variables for which the denominator of the fraction is 0. Since dividing by 0 is not allowed, you need to avoid using such values of the variables! For example, the formula above for the force F due to gravity cannot be applied with $r = 0$. In fact, r takes only positive values, since it represents a distance between two different objects.

4.1 Equivalent algebraic fractions

Recall from Section 3 of Unit 3 that two numerical fractions are **equivalent** if one can be obtained from the other by multiplying or dividing both the numerator and the denominator by the same number. For example,

$$\frac{2}{6} \text{ is equivalent to } \frac{1}{3}, \quad \text{because} \quad \frac{2}{6} = \frac{1 \times 2}{3 \times 2}.$$

Similarly, two algebraic fractions are equivalent if one can be obtained from the other by multiplying or dividing both the numerator and the denominator by the same expression. For example,

$$\frac{a}{b} \text{ is equivalent to } \frac{a(a+1)}{b(a+1)}.$$

However, notice that the first expression here is valid for $a = -1$, whereas the second expression is not.

The process of simplifying a numerical fraction by dividing both the numerator and the denominator by the same number is called **cancelling** a common factor. For example, the fraction $\frac{2}{6}$ can be simplified by cancelling the common factor 2:

$$\frac{2}{6} = \frac{\overset{1}{\cancel{2}}}{\underset{3}{\cancel{6}}} = \frac{1}{3}.$$

Algebraic fractions can be simplified in the same way, by cancelling any common factors of the numerator and denominator. You can represent the cancellation in the usual way, by crossing out the expressions with common factors and writing the results of the cancellations nearby.

This crossing out is not always shown, however.

You can find any common factors of the numerator and denominator of an algebraic fraction by expressing the numerator and denominator as products of all their factors, if this is not done already. Here are some examples.

Tutorial clip

Example 11 *Simplifying algebraic fractions*

Simplify each of the following algebraic fractions.

(a) $\dfrac{a^2}{a^5}$ (b) $\dfrac{60p^3q}{35p^5r}$ (c) $\dfrac{2x^2+6x}{x^2-9}$

Solution

(a) Divide top and bottom by a^2.

$$\frac{a^2}{a^5} = \frac{\overset{1}{\cancel{a^2}}}{\underset{a^3}{\cancel{a^5}}} = \frac{1}{a^3}$$

Now top and bottom have no common factors, so the fraction can't be simplified further.

(b) Divide top and bottom by 5 and p^3.

$$\frac{60p^3q}{35p^5r} = \frac{\overset{12}{\cancel{60}}\,\overset{1}{\cancel{p^3}}\,q}{\underset{7}{\cancel{35}}\,\underset{p^2}{\cancel{p^5}}\,r} = \frac{12q}{7p^2r}$$

(c) To find any common factors, factorise the top and bottom.

The denominator x^2-9 is a difference of two squares.

$$\frac{2x^2+6x}{x^2-9} = \frac{2x(x+3)}{(x-3)(x+3)} = \frac{2x\overset{1}{\cancel{(x+3)}}}{(x-3)\underset{1}{\cancel{(x+3)}}} = \frac{2x}{x-3}$$

In Example 11, the expressions that are factors of the denominators are all assumed to be non-zero. For example, in part (a) it is assumed that $a \neq 0$. Similarly, in part (c) it is assumed that $x-3 \neq 0$ and $x+3 \neq 0$. Often, assumptions of this type are not stated explicitly, but you should be aware of them.

It is important to cancel only expressions that are *factors* of both the numerator and the denominator of a fraction. For example, in the fraction

$$\frac{2x^2+6x}{x^2-9},$$

in Example 11, it would have been wrong to cancel the expression x^2 in the numerator and denominator, because x^2 is not a factor of the numerator or denominator. So before you cancel anything, check that it is a factor of both the top and the bottom!

Here are some algebraic fractions for you to simplify.

Activity 27 *Simplifying algebraic fractions*

In part (d), you need to factorise both the numerator and the denominator.

Simplify the following algebraic fractions.

(a) $\dfrac{x^4}{x^9}$ (b) $\dfrac{20a^2b}{15ab^2}$ (c) $\dfrac{x^2+6x}{3x^2}$ (d) $\dfrac{u^2-4}{u^2+4u+4}$

4.2 Adding and subtracting algebraic fractions

The rules for adding and subtracting algebraic fractions are the same as those for adding and subtracting numerical fractions, which were given in Section 2 of Unit 3.

Here are some numerical examples. If the denominators are the same, then you just add or subtract the numerators:

$$\frac{3}{5} - \frac{2}{5} = \frac{1}{5}.$$

If the denominators are different, then you need to write the fractions with a common denominator before you can add or subtract them. For example, $\frac{2}{5}$ and $\frac{3}{8}$ can be added or subtracted by using the common denominator $5 \times 8 = 40$, as follows:

$$\frac{2}{5} + \frac{3}{8} = \frac{2 \times 8}{5 \times 8} + \frac{3 \times 5}{8 \times 5} = \frac{16}{40} + \frac{15}{40} = \frac{31}{40},$$

$$\frac{2}{5} - \frac{3}{8} = \frac{2 \times 8}{5 \times 8} - \frac{3 \times 5}{8 \times 5} = \frac{16}{40} - \frac{15}{40} = \frac{1}{40}.$$

The idea of finding a common denominator works for algebraic fractions too – the general strategy is given below.

Strategy *To add or subtract algebraic fractions*

1. Make sure that the fractions have a common denominator – if necessary, rewrite each fraction as an equivalent fraction.
2. Add or subtract the numerators.
3. Simplify the answer by cancelling wherever possible.

Here are some examples to illustrate this strategy.

Example 12 *Adding and subtracting algebraic fractions*

Tutorial clip

Write each of the following expressions as a single algebraic fraction.

(a) $\dfrac{3}{x} - \dfrac{2}{x}$ (b) $\dfrac{3}{a+1} + \dfrac{2}{a+1} - \dfrac{1}{a+1}$ (c) $\dfrac{a}{x} + \dfrac{b}{y}$

Solution

(a) The denominators are the same, so subtract the numerators.

$$\frac{3}{x} - \frac{2}{x} = \frac{3-2}{x} = \frac{1}{x}$$

(b) The denominators are the same, so add and subtract the numerators.

$$\frac{3}{a+1} + \frac{2}{a+1} - \frac{1}{a+1} = \frac{3+2-1}{a+1} = \frac{4}{a+1}$$

(c) The product of the denominators is xy, so use this as a common denominator.

$$\frac{a}{x} + \frac{b}{y} = \frac{ay}{xy} + \frac{bx}{xy} = \frac{ay+bx}{xy}$$

As illustrated in Example 12(c), when you want to add or subtract algebraic fractions that do not have a common denominator, you can

always obtain a common denominator by multiplying together the denominators of the given fractions.

Activity 28 *Adding and subtracting algebraic fractions*

Write each of the following expressions as a single algebraic fraction.

(a) $\dfrac{6}{y}+\dfrac{1}{y}$ (b) $\dfrac{x}{a^2}+\dfrac{y}{a^2}-\dfrac{z}{a^2}$ (c) $\dfrac{a}{2x}-\dfrac{b}{3y}$ (d) $\dfrac{1}{3x}-\dfrac{2}{x+3}$

Although you can always obtain a common denominator by multiplying together the denominators of the given fractions, there is sometimes a simpler common denominator. In the example

$$\frac{2}{a}+\frac{1}{a^2},$$

the product of the denominators is $a\times a^2=a^3$. But a simpler common denominator is a^2, because a^2 is a multiple of both a and a^2. Using this common denominator gives

$$\frac{2}{a}+\frac{1}{a^2}=\frac{2a}{a^2}+\frac{1}{a^2}=\frac{2a+1}{a^2}.$$

If you had used the common denominator a^3, then the result would have been the same, but only after cancelling a common factor of a from the numerator and the denominator.

To find the simplest common denominator, you should factorise the denominators of the given fractions, if possible, and then choose the simplest expression that is a multiple of each denominator. For example, to write

$$\frac{x}{xy+y^2}+\frac{y}{x^2+xy}$$

as a single algebraic fraction, you should factorise the denominators to give

$$\frac{x}{y(x+y)}+\frac{y}{x(x+y)}$$

and then choose as your common denominator $xy(x+y)$. So

$$\begin{aligned}\frac{x}{xy+y^2}+\frac{y}{x^2+xy}&=\frac{x}{y(x+y)}+\frac{y}{x(x+y)}\\&=\frac{x^2}{xy(x+y)}+\frac{y^2}{xy(x+y)}\\&=\frac{x^2+y^2}{xy(x+y)}.\end{aligned}$$

Also, when working with fractions you sometimes need to make an expression into a fraction in order to combine it with another fraction. For example, to express

$$a+\frac{2}{b}$$

as a single fraction, you can write a as the fraction $a/1$, and then choose the common denominator to be b:

$$a+\frac{2}{b}=\frac{a}{1}+\frac{2}{b}=\frac{ab}{b}+\frac{2}{b}=\frac{ab+2}{b}.$$

Activity 29 *Choosing denominators*

Write each of the following expressions as a single algebraic fraction.

(a) $\dfrac{5}{x^4} - \dfrac{4}{x^2}$ (b) $\dfrac{1}{x^2+x} - \dfrac{1}{x+1}$ (c) $a + \dfrac{a}{b+3}$

4.3 Multiplying and dividing algebraic fractions

The rules for multiplying and dividing algebraic fractions are the same as those for multiplying and dividing numerical fractions, which were given in Section 2 of Unit 3.

Here are some numerical examples. To multiply two numerical fractions, multiply the numerators together and multiply the denominators together:

$$\frac{1}{5} \times \frac{3}{5} = \frac{1\times 3}{5\times 5} = \frac{3}{25}.$$

To divide by a numerical fraction, multiply by its reciprocal:

$$\frac{1}{5} \div \frac{3}{5} = \frac{1}{5} \times \frac{5}{3} = \frac{1 \times \overset{1}{\cancel{5}}}{\underset{1}{\cancel{5}} \times 3} = \frac{1}{3}.$$

Remember that to obtain the reciprocal of a fraction, you swap the numerator and denominator; in other words, you 'turn the fraction upside down'.

Strategy *To multiply or divide algebraic fractions*

- To multiply two algebraic fractions, multiply the numerators together and multiply the denominators together:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

- To divide one algebraic fraction by another, multiply the first fraction by the reciprocal of the second fraction:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

In each case, you should cancel any common factors that appear.

The following examples illustrate this strategy.

Example 13 *Multiplying algebraic fractions*

Tutorial clip

Write each of the following expressions as a single algebraic fraction, simplifying your answer if possible.

(a) $\dfrac{a}{2x} \times \dfrac{b}{3y}$ (b) $\dfrac{3x}{5y} \times \dfrac{y(y+1)}{x^2}$

Solution

(a) $$\frac{a}{2x} \times \frac{b}{3y} = \frac{a \times b}{2x \times 3y} = \frac{ab}{6xy}$$

(b) $$\frac{3x}{5y} \times \frac{y(y+1)}{x^2} = \frac{3x \times y(y+1)}{5y \times x^2} = \frac{3\overset{1}{\cancel{x}} \times \overset{1}{\cancel{y}}(y+1)}{5\underset{1}{\cancel{y}} \times \underset{x}{\cancel{x^2}}} = \frac{3(y+1)}{5x}$$

Tutorial clip

Example 14 *Dividing algebraic fractions*

Write each of the following expressions as a single algebraic fraction, simplifying your answer if possible.

(a) $\dfrac{a}{2x} \div \dfrac{b}{3y}$ (b) $\dfrac{3x}{5y} \div \dfrac{y(y+1)}{x^2}$

Solution

(a) $\dfrac{a}{2x} \div \dfrac{b}{3y} = \dfrac{a}{2x} \times \dfrac{3y}{b} = \dfrac{a \times 3y}{2x \times b} = \dfrac{3ay}{2bx}$

(b) $\dfrac{3x}{5y} \div \dfrac{y(y+1)}{x^2} = \dfrac{3x}{5y} \times \dfrac{x^2}{y(y+1)} = \dfrac{3x \times x^2}{5y \times y(y+1)} = \dfrac{3x^3}{5y^2(y+1)}$

There are also some special cases of multiplying and dividing fractions. These arise when one of the expressions being multiplied or divided is not written as a fraction.

For example, you can do the following multiplication and division by writing $3b^2$ as a fraction with denominator 1:

$$3b^2 \times \frac{1}{a} = \frac{3b^2}{1} \times \frac{1}{a} = \frac{3b^2}{a}$$

and

$$\frac{b}{b+1} \div 3b^2 = \frac{b}{b+1} \div \frac{3b^2}{1} = \frac{b}{b+1} \times \frac{1}{3b^2} = \frac{1}{3b(b+1)}.$$

Here and from now on, the crossing out for any cancellation is not shown, but you may choose to include it if you find it helpful.

Here are some examples of multiplying and dividing algebraic fractions for you to try.

Activity 30 *Multiplying and dividing fractions*

Write each of the following expressions as a single algebraic fraction, simplifying your answer if possible.

(a) $\dfrac{p^2}{q^2} \times \dfrac{p}{q}$ (b) $\dfrac{p^2}{q^2} \div \dfrac{p}{q}$ (c) $\dfrac{9ax^2}{b} \times \dfrac{b^3}{4xy^2}$

(d) $\dfrac{9ax^2}{b} \div \dfrac{b^3}{4xy^2}$ (e) $8u^2 \times \dfrac{2}{u^2+u}$

To end this section, here is a practical application of dividing fractions.

Activity 31 *Finding the ratio of two forces*

In this activity, the mass of the Earth is M kg and its radius is R km, and the mass of the Moon is m kg and its radius is r km. The gravitational force acting on an object of mass 1 kg is given by the formulas

$$F = \frac{GM}{R^2} \quad \text{on the surface of the Earth}$$

and

$$f = \frac{Gm}{r^2} \quad \text{on the surface of the Moon,}$$

where G is a constant.

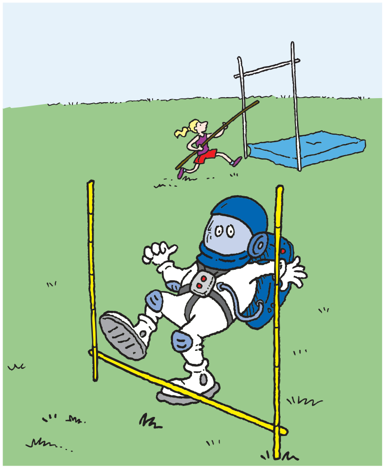
The actual values, to two significant figures, are as follows:
$M = 6.0 \times 10^{24}$,
$R = 6400$,
$m = 7.4 \times 10^{22}$,
$r = 1700$.

(a) Express the ratio F/f as an algebraic fraction involving M, m, R and r.

(b) Assuming that $M = 80m$ and $R = 4r$, deduce that $F = 5f$.

The data above show that these equations are approximately correct.

Part (b) of Activity 31 shows that the gravitational force acting on an object on the surface of the Earth is approximately five times greater than the gravitational force acting on the same object on the surface of the Moon. One dramatic consequence of this fact is that an astronaut can jump five times higher on the Moon than on the Earth! However, for a direct comparison the astronaut would have to wear the same spacesuit on Earth as on the Moon.



5 Rearranging formulas

Many branches of science are concerned with the relationship between different variables that are connected together in some way. For example, in physics the equation

$$PV = kT$$

relates the pressure P, volume V and temperature T of an enclosed quantity of a gas. Here k is a constant. In the form stated above, this equation does not have a subject (a single variable that appears only once, on its own on the left-hand side), but the formula can easily be rearranged to make P, for example, the subject:

The first version of this equation was published in 1662 by the Irish scientist Robert Boyle.

$$P = \frac{kT}{V}.$$

From this formula, you can see that if the temperature of the gas remains the same but the volume decreases, then the pressure of the gas increases.

The next subsection shows you how to solve equations by rearranging algebraic fractions. The final two subsections show you how to rearrange formulas that involve algebraic fractions or powers.

5.1 Solving equations by clearing algebraic fractions

In Section 5 of Unit 5 you saw a strategy for solving linear equations. This involved carrying out a sequence of steps of the following types:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

You saw that the strategy can be applied to solve equations such as

$$3\left(1+\frac{x}{2}\right) = 2(x-1). \qquad (5)$$

The first step is to multiply both sides of the equation by 2 to clear the fraction on the left. Then you multiply out the brackets and rearrange the terms so that the unknown x is left by itself on one side of the equation. The complete sequence of steps is as follows:

Multiply by 2:	$6\left(1+\frac{x}{2}\right) = 4(x-1)$
Multiply out the brackets:	$6+3x = 4x-4$
Subtract $3x$ and add 4:	$10 = x$

This gives the answer $x = 10$, and you can check that this value of x satisfies equation (5).

An equation involving algebraic fractions can be solved in a similar way. Usually the best first step is to clear the fractions, and this can be done by multiplying both sides of the equation by an expression that is a multiple of all the denominators, as illustrated in the next example.

Example 15 *Solving an equation by clearing algebraic fractions*

Solve the equation

$$\frac{2}{x} = \frac{1}{x-1}.$$

Solution

If $x = 0$ or $x = 1$, then one of the fractions is undefined.

Assume that $x \neq 0$ and $x \neq 1$.

Multiply by $x(x-1)$, which is a multiple of both x and $x-1$.

Multiply by $x(x-1)$:	$x(x-1)\frac{2}{x} = x(x-1)\frac{1}{x-1}$
Cancel:	$2(x-1) = x$
Multiply out:	$2x-2 = x$
Rearrange:	$x = 2$

The value $x = 2$ satisfies the assumptions, so it is the solution.

(Check: When $x = 2$,

$$\text{LHS} = \frac{2}{2} = 1 \quad \text{and} \quad \text{RHS} = \frac{1}{2-1} = 1.)$$

In the next example, clearing the algebraic fractions gives a quadratic equation, which you can factorise using the methods of Section 3.

Example 16 *Solving another equation by clearing algebraic fractions*

Solve the equation

$$\frac{2}{x} + 3 = \frac{x+10}{4}.$$

Solution

Assume that $x \neq 0$.

Multiply by $4x$, which is a multiple of both x and 4. Use brackets to show that the *whole* of each side is multiplied by $4x$.

Multiply by $4x$:	$4x\left(\frac{2}{x}+3\right) = 4x\left(\frac{x+10}{4}\right)$
Cancel on the RHS:	$4x\left(\frac{2}{x}+3\right) = x(x+10)$
Multiply out both sides:	$8 + 12x = x^2 + 10x$
Subtract $8 + 12x$ and swap sides:	$x^2 - 2x - 8 = 0$
Factorise:	$(x-4)(x+2) = 0$

This gives $x = 4$ or $x = -2$, and these values satisfy the assumptions, so they are the solutions.

(Check: When $x = 4$,

$$\text{LHS} = \frac{2}{4} + 3 = 3\tfrac{1}{2} \quad \text{and} \quad \text{RHS} = \frac{4+10}{4} = 3\tfrac{1}{2}.$$

When $x = -2$,

$$\text{LHS} = \frac{2}{-2} + 3 = -1 + 3 = 2 \quad \text{and} \quad \text{RHS} = \frac{-2+10}{4} = \frac{8}{4} = 2.)$$

Here are some equations involving algebraic fractions for you to solve.

Activity 32 *Solving equations involving algebraic fractions*

Solve each of the following equations.

(a) $\dfrac{4}{x-3} = \dfrac{12}{x+1}$ (b) $\dfrac{t}{5} = \dfrac{2}{t+3}$

(c) $\dfrac{4}{x} + 2x = 9$ (d) $\dfrac{10}{x} + \dfrac{20}{x+40} = \dfrac{1}{2}$

There is a technique called *cross-multiplying* that can cut down a little of the work when you rearrange equations like those in parts (a) and (b) of this activity. You can learn about it in Unit 14.

In the two examples in this subsection, assumptions were made about the value of the unknown, to ensure that the denominators of the fractions were non-zero. These assumptions also ensured that the expressions by which both sides of the equations were multiplied were non-zero. In general, when you are trying to solve an equation, you should not multiply both sides by an expression that might be zero. To see why, consider the simple equation

$$x - 1 = 0. \tag{6}$$

This equation has just one solution, namely $x = 1$. Now look at what happens when you multiply both sides by the expression $x - 2$.

You obtain the equation

$$(x-1)(x-2)=0,$$

which has *two* solutions, $x=1$ and $x=2$. So multiplying an equation by an expression can generate extra solutions that were not solutions of the original equation.

You can get round this problem by assuming that any expression that you multiply by is non-zero. Then, at the end of your working, you should check that the solutions that you have obtained satisfy the assumptions that you made, and disregard any solutions that do not. For example, you can multiply equation (6) by the expression $x-2$ *provided that you assume that* $x \neq 2$. At the end of your working, you would disregard the solution $x=2$, leaving you with just the correct solution $x=1$.

Multiplying by $x-2$ would not be a sensible way to solve equation (6), of course!

5.2 Rearranging formulas by clearing algebraic fractions

In Section 2 of Unit 7 you saw how to rearrange an equation to make a chosen variable the subject, by using the following strategy. This strategy is closely related to the techniques used in the previous subsection to solve equations.

Strategy *To make a variable the subject of an equation*

Carry out a sequence of steps. In each step, do one of the following:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

Aim to do the following, in order.

1. Clear any fractions and multiply out any brackets. To clear fractions, multiply both sides by a suitable expression.
2. Add or subtract terms on both sides to get all the terms containing the required subject on one side, and all the other terms on the other side.
3. If more than one term contains the required subject, then take it out as a common factor. This gives an equation of the form

   $$\text{an expression} \times \text{the required subject} = \text{an expression}.$$

4. Divide both sides by the expression that multiplies the required subject.

Here is an example of this strategy in action, from Section 2 of Unit 7.

Suppose that you are asked to make s the subject of the equation

$$r=\frac{s}{r}+2s.$$

Multiply by r (assuming that $r \neq 0$):	$r^2=r\left(\frac{s}{r}+2s\right)$
Multiply out:	$r^2=s+2rs$
Take out s as a common factor:	$r^2=s(1+2r)$

Divide by $1+2r$ (assuming that $1+2r \neq 0$): $\dfrac{r^2}{1+2r} = s$

Swap the sides: $s = \dfrac{r^2}{1+2r}$

The first two steps in this list clear the fraction s/r in the original equation.

Let's look at an example that is a little more complicated than the examples that you saw in Unit 7: it involves more than one algebraic fraction.

In optics, the *lens formula*,

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v},$$

relates:

- the *focal length* f of a lens, that is, the distance from the lens at which (distant) parallel rays of light are focused
- the *object distance* u, that is, the distance from the lens of the object being viewed
- the *image distance* v, that is, the distance from the lens of the image of the object.

Strictly speaking, the lens formula is not a formula in the sense defined in Unit 2 (since it does not have a subject – a variable that appears only once, on its own on one side), but the word 'formula' is sometimes used loosely in mathematics.

These distances are illustrated in Figure 10. The schematic diagram in part (a) of the figure shows a simple demonstration of how a lens can be used to project the image of a candle onto a screen. Five representative light rays are shown travelling from the candle, through the lens, to form the image on the screen. With this arrangement, the resulting image appears inverted.

Figure 10(b) shows the associated *ray diagram* for this scenario. Here you can see the object distance u, image distance v and focal length f, which are related by the lens formula.

Ray diagrams are used in optics to describe how lenses can form images of varying size and shape, depending on the type of lens and how they are focused.

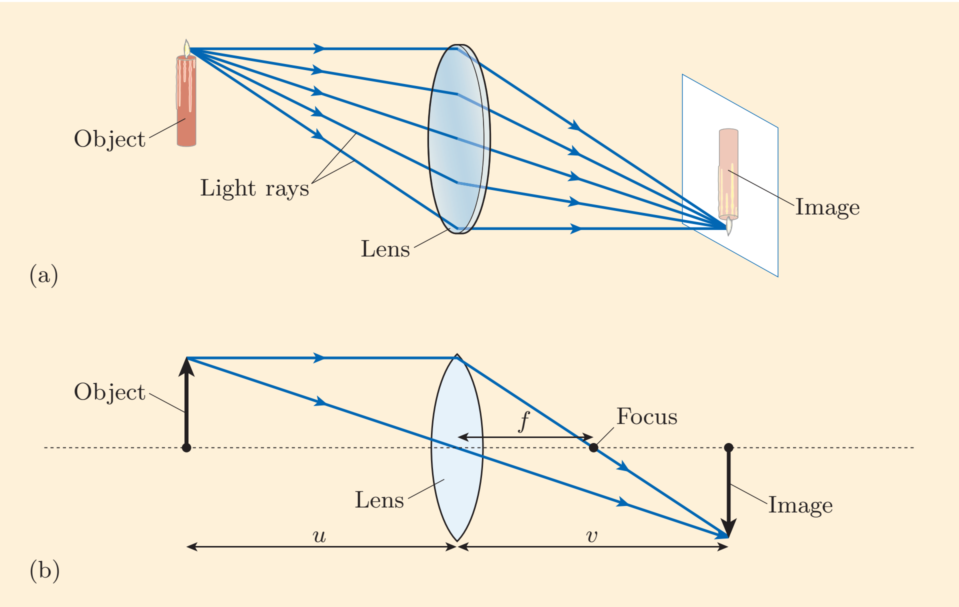


Figure 10 (a) Image formed by projecting an object through a convex lens. (b) Ray diagram of the scenario illustrated in (a).

The lens formula plays a fundamental role in optics; for example, it is used in the design of lenses for correcting eye disorders such as long-sightedness and short-sightedness.

The example below involves making a particular variable the subject of the lens formula.

Tutorial clip

Example 17 *Rearranging the lens formula*

Make u the subject of the lens formula

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}.$$

Solution

To clear all the fractions, multiply both sides by f, u and v. Assume that $f \neq 0$, $u \neq 0$ and $v \neq 0$.

Multiply both sides by the product fuv:

$$\frac{fuv}{f} = \frac{fuv}{u} + \frac{fuv}{v}.$$

Cancel:

$$uv = vf + uf.$$

Get all the terms containing the required subject u on the left and all other terms on the right.

Rearrange the equation:

$$uv - uf = vf.$$

Take out u as a common factor:

$$u(v - f) = vf.$$

Now assume that $v - f \neq 0$. Then you can divide both sides by $v - f$ to give u as the subject:

$$u = \frac{vf}{v - f}.$$

You can make v or f the subject of the lens formula in a similar way:

$$v = \frac{uf}{u - f}, \quad \text{assuming that } u - f \neq 0,$$

and

$$f = \frac{uv}{u + v}, \quad \text{assuming that } u + v \neq 0.$$

Here are some rearrangements of this type for you to try.

Activity 33 *Changing the subject*

(a) Make u the subject of the equation $v = \dfrac{2}{u + 3} + 4$.

(b) Make x the subject of the equation $y = \dfrac{20}{3x} - \dfrac{3}{2}$.

As you saw in Unit 7, there is often more than one way to rearrange a formula to make a new variable the subject. For example in Activity 33(a), you could have begun by subtracting 4 from both sides:

$$v - 4 = \frac{2}{u+3}.$$

Then if you multiply by $u + 3$ to clear the fraction, and divide by $v - 4$ (assuming that $u + 3 \neq 0$ and $v - 4 \neq 0$), you obtain

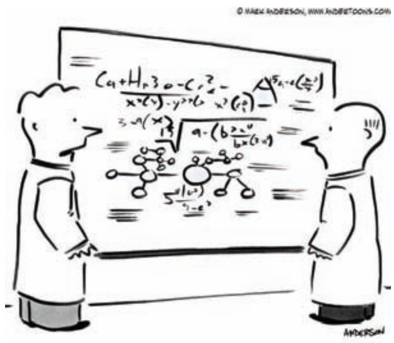
$$u + 3 = \frac{2}{v-4}.$$

Subtracting 3 and then putting the right-hand side over a common denominator gives

$$u = \frac{2}{v-4} - 3 = \frac{2}{v-4} - \frac{3(v-4)}{v-4} = \frac{2-3(v-4)}{v-4} = \frac{14-3v}{v-4}.$$

5.3 Formulas involving powers

In this last subsection, you will see how to change the subject of a formula involving powers. This involves using several properties of powers that were given in Unit 3.



"I don't know, it's a little formulaic."

For example, the formula

$$A = \pi r^2$$

expresses the area A of a circle in terms of its radius r and the constant $\pi = 3.141\,59\ldots$. This formula involves the power r^2.

Suppose that you want to rearrange this formula to make r the subject; that is, to get r by itself, just on the left-hand side of the equation. Once again you have to do this by carrying out a sequence of steps of the following types:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

You can use steps of these types to get r^2 on its own on the left-hand side, as follows.

Divide by π: $\quad \dfrac{A}{\pi} = r^2$

Swap the sides: $\quad r^2 = \dfrac{A}{\pi}$

We can now get r on its own by taking the square root of both sides:

$$r = \pm\sqrt{\frac{A}{\pi}}.$$

We can apply the square root operation to the right-hand side because the expression A/π is positive (because A is an area).

Since r is a length, it is positive or zero, so the negative square root can be disregarded. Hence the formula is

$$r = \sqrt{\frac{A}{\pi}},$$

in which r is the subject.

See Unit 3 for a discussion of square roots and other roots.

The formula for r, obtained above, can be rewritten in various forms, such as

$$r = \left(\frac{A}{\pi}\right)^{\frac{1}{2}}, \quad r = \frac{A^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \quad \text{and} \quad r = \frac{\sqrt{A}}{\sqrt{\pi}}.$$

These forms are obtained by using the fact that taking the square root of a positive number is the same as raising the number to the power $\frac{1}{2}$, together with the rule $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$.

In making r the subject of this formula, we used the same strategy as before, but here we applied a new operation to both sides of the equation, namely taking the square root of both sides, which is allowed only if both sides of the equation are positive.

Just as raising to the power $\frac{1}{2}$ is the reverse of squaring, so raising to the power $\frac{1}{3}$ is the reverse of raising to the power 3, and so on. We can extend the list of operations that we can apply to both sides of an equation, to include raising to a power, as long as we only apply this operation to *positive* quantities.

Thus if you can rearrange an equation into the form

$$(\text{the required subject})^{\text{a power}} = \text{an expression},$$

where the expression on the right-hand side is positive, then you can obtain the required subject on its own on the left-hand side by raising both sides to the reciprocal of the power that the required subject is raised to.

Here is an example.

Tutorial clip

Example 18 *Rearranging an equation with powers*

The volume V of a sphere of radius r is given by the formula

$$V = \tfrac{4}{3}\pi r^3.$$

Make r the subject of this formula.

Solution

Multiply by 3: $\quad 3V = 4\pi r^3$

Divide by 4π: $\quad \dfrac{3V}{4\pi} = r^3$

Here, the expression $\dfrac{3V}{4\pi}$ is positive because V is a volume.

Swap the sides: $\quad r^3 = \dfrac{3V}{4\pi}$

Raise to the power $\frac{1}{3}$: $\quad r = \left(\dfrac{3V}{4\pi}\right)^{\frac{1}{3}}$

Here are some similar rearrangements for you to try.

Activity 34 *Rearranging equations with powers*

(a) The volume V of a cone whose base has radius r and whose height is h (Figure 11) is given by the formula

$$V = \tfrac{1}{3}\pi r^2 h.$$

(i) Make r the subject of this formula.

(ii) Hence find, to three significant figures, the base radius of a cone whose height is $1\,\mathrm{m}$ and whose volume is $2\,\mathrm{m}^3$.

(b) The *Tressider formula* was used in the nineteenth century for designing a ship's armour plating. It is

$$T^2 = \frac{MS^3}{cD},$$

where T is the required thickness of the armour plating, S is the speed of a shell fired at the ship, D is its diameter and M is its mass, and c is a constant.

Make S the subject of this formula.

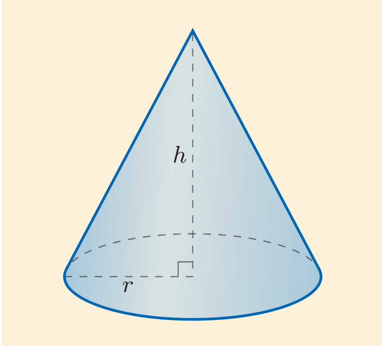


Figure 11 A cone

The key thing to remember when rearranging equations and formulas is always to do the same operation to both sides!

Learning checklist

After studying this unit, you should be able to:

- find the sum of any arithmetic sequence
- prove simple number patterns involving square numbers
- multiply out pairs of brackets
- solve quadratic equations by using factorisation, where possible
- add, subtract, multiply and divide algebraic fractions
- solve certain equations involving algebraic fractions
- rearrange certain formulas involving algebraic fractions and powers.

Solutions and comments on Activities

Activity 1

We can rearrange the numbers in the sum in pairs as follows:

$$1 + 2 + \cdots + 199 + 200$$
$$= (1 + 200) + (2 + 199) + \cdots + (100 + 101).$$

There are 100 pairs of numbers in brackets, and each pair has sum 201, so

$$1 + 2 + \cdots + 199 + 200 = 100 \times 201 = 20\,100.$$

Activity 2

Denote the sum of the first n natural numbers by S. Then

$$S = 1 + 2 + \cdots + (n-1) + n$$

and, in reverse order,

$$S = n + (n-1) + \cdots + 2 + 1.$$

By adding these two equations and rearranging the right-hand side, you obtain

$$2S = (1+n) + (2 + (n-1)) + \cdots + ((n-1) + 2) + (n+1).$$

In this sum there are n pairs of numbers altogether, each pair having sum $n + 1$. Thus

$$2S = n \times (n+1), \quad \text{so} \quad S = \tfrac{1}{2}n(n+1).$$

Activity 3

(a) $3, 10, 17, 24, 31$

(b) $3, -2, -7, -12, -17, -22, -27, -32$

Activity 4

(a) One trolley has length 75 cm. Two stacked trolleys have length

$$75\,\text{cm} + 15\,\text{cm} = 90\,\text{cm}.$$

Three stacked trolleys have length

$$90\,\text{cm} + 15\,\text{cm} = 105\,\text{cm}.$$

(b) These lengths form the arithmetic sequence

$$75,\ 90,\ 105,\ \ldots.$$

So $a = 75$ and $d = 15$.

(c) The length of 20 stacked trolleys (in centimetres) is given by the 20th term in this sequence. By equation (1), the 20th term is

$$75 + (20-1) \times 15 = 75 + 285 = 360.$$

So the length of 20 stacked trolleys is 3.6 m.

Activity 5

The equation is:	$L = a + (n-1)d$
Subtract a:	$L - a = (n-1)d$
Swap sides:	$(n-1)d = L - a$
Divide by d:	$n - 1 = \dfrac{L-a}{d}$
Add 1:	$n = \dfrac{L-a}{d} + 1$

Activity 6

(a) This arithmetic sequence has first term $a = 2$, difference $d = 3$ and last term $L = 29$. The number of terms n of the sequence is given by

$$n = \frac{29-2}{3} + 1 = 9 + 1 = 10.$$

(Check: The sequence is $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$, which has 10 terms.)

(b) From part (a) the number of terms is 10, so we use the formula with $n = 10$:

$$S = \tfrac{1}{2}n(2a + (n-1)d)$$
$$= \tfrac{1}{2} \times 10(4 + 9 \times 3) = 155.$$

So the sum of the sequence is 155.

(Alternatively you can use the formula involving the last term:

$$S = \tfrac{1}{2}n(a + L) = \tfrac{1}{2} \times 10 \times (2 + 29) = 5 \times 31 = 155.)$$

Activity 7

The first salary scale is an arithmetic sequence with first term $a = 20\,000$, difference $d = 500$ and number of terms $n = 10$. Thus the total amount paid over 10 years, in pounds, is

$$S = \tfrac{1}{2}n(2a + (n-1)d)$$
$$= \tfrac{1}{2} \times 10 \times (2 \times 20\,000 + 9 \times 500)$$
$$= 5 \times (40\,000 + 4500) = 222\,500.$$

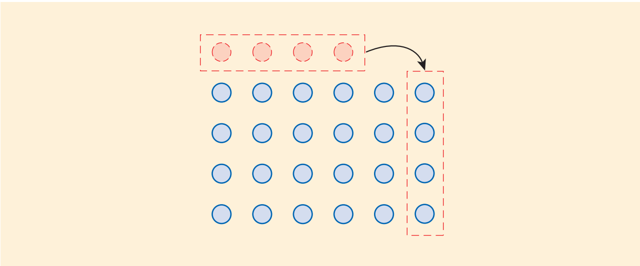
The second salary scale is an arithmetic sequence with first term $a = 18\,000$, difference $d = 1000$ and number of terms $n = 10$. Thus the total amount paid over 10 years, in pounds, is

$$S = \tfrac{1}{2}n(2a + (n-1)d)$$
$$= \tfrac{1}{2} \times 10 \times (2 \times 18\,000 + 9 \times 1000)$$
$$= 5 \times (36\,000 + 9000) = 225\,000.$$

Therefore the second job pays (slightly) more over 10 years.

Activity 8

(a) After the dot in the top right corner is removed, the four dots remaining in the top row can be moved to one of the sides to make a 4×6 rectangle of dots.



Before the four dots were moved there were $5^2 - 1$ dots, and afterwards there are 4×6 dots. This illustrates the equation $5^2 - 1 = 4 \times 6$.

(b) Consider a square consisting of n^2 dots. This can be converted into a rectangle consisting of $(n-1)(n+1)$ dots by removing the top right corner dot and moving a line of $n - 1$ dots from the top row to the right side. Hence

$$n^2 - 1 = (n-1)(n+1).$$

Activity 9

(a) $a(2b + 3c) = 2ab + 3ac$

(b) $-r(2s - 3t) = -2rs + 3rt$

(c) $(n-1)n = n^2 - n$

Activity 10

(a) $$\begin{aligned}(x+2)(x+4) &= x^2 + 4x + 2x + 8\\ &= x^2 + 6x + 8\end{aligned}$$

(b) $(a + 2b)(3x + 4y) = 3ax + 4ay + 6bx + 8by$

(c) $$\begin{aligned}(x-3)^2 &= (x-3)(x-3)\\ &= x^2 - 3x - 3x + 9\\ &= x^2 - 6x + 9\end{aligned}$$

(d) $(a - 2b)(3c - d) = 3ac - ad - 6bc + 2bd$

(e) $(p-1)(-2+3q) = -2p + 3pq + 2 - 3q$

(f) $(n-2)(n+2) = n^2 + 2n - 2n - 4 = n^2 - 4$

Activity 11

$$\begin{aligned}&(2a-b)(c-3d+2e)\\ &= 2a(c-3d+2e) - b(c-3d+2e)\\ &= 2ac - 6ad + 4ae - bc + 3bd - 2be\end{aligned}$$

Activity 12

(a) Writing down the answer directly gives

$$(x+7)^2 = x^2 + 14x + 49.$$

(b) Writing down the answer directly gives

$$(u-10)^2 = u^2 - 20u + 100.$$

(c) Writing down the answer directly gives

$$\begin{aligned}(t+\tfrac{1}{2})^2 &= t^2 + 2 \times \tfrac{1}{2}t + (\tfrac{1}{2})^2\\ &= t^2 + t + \tfrac{1}{4}.\end{aligned}$$

(d) Multiplying out the brackets gives

$$\begin{aligned}(3x+2)^2 &= (3x+2)(3x+2)\\ &= (3x)^2 + 6x + 6x + 2^2\\ &= 9x^2 + 12x + 4.\end{aligned}$$

(e) Multiplying out the brackets:

$$\begin{aligned}(2s-5t)^2 &= (2s-5t)(2s-5t)\\ &= (2s)^2 - 10st - 10st + (5t)^2\\ &= 4s^2 - 20st + 25t^2.\end{aligned}$$

Activity 13

(a) By the difference of two squares identity,

$$(u-12)(u+12) = u^2 - 12^2 = u^2 - 144.$$

(b) By the difference of two squares identity,

$$(x-2y)(x+2y) = x^2 - (2y)^2 = x^2 - 4y^2.$$

(c) By the difference of two squares identity,

$$(10-a)(10+a) = 10^2 - a^2 = 100 - a^2.$$

Activity 14

(a) $9x^2 - 12x + 4$ is a quadratic, with $a = 9$, $b = -12$ and $c = 4$.

(b) $3y - 5$ is not a quadratic, as it has no squared term.

(c) $-6 - 7s^2$ is a quadratic, with $a = -7$, $b = 0$ and $c = -6$.

(d) $2x^3 + x^2$ is not a quadratic, as it has a term in x^3.

Activity 15

Substituting $x = 2$ in the quadratic expression gives

$$\begin{aligned}2x^2 - x - 6 &= 2 \times 2^2 - 2 - 6\\ &= 8 - 2 - 6 = 0.\end{aligned}$$

Hence $x = 2$ is a solution of the equation.

Activity 16

(a) This equation can be rearranged as $x^2 = 25$, so it has two solutions, $x = \pm 5$.

(b) This equation can be rearranged as $t^2 = 3$, so it has two solutions, $t = \pm\sqrt{3}$.

Activity 17

(a) The quadratic is $x^2 + 3x + 2$.

The pair $1, 2$ has sum 3 and product 2. Thus

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x + 1)(x + 2) &= x^2 + 2x + x + 2\\ &= x^2 + 3x + 2.)\end{aligned}$$

(b) The quadratic is $x^2 + 11x + 24$.

The positive factor pairs of 24 are

$1, 24, \quad 2, 12, \quad 3, 8, \quad 4, 6.$

The only pair whose sum is 11 is $3, 8$. Thus

$$x^2 + 11x + 24 = (x + 3)(x + 8).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x + 3)(x + 8) &= x^2 + 8x + 3x + 24\\ &= x^2 + 11x + 24.)\end{aligned}$$

Activity 18

(a) The quadratic is $x^2 - 10x + 24$.

The negative factor pairs of 24 are

$-1, -24, \quad -2, -12, \quad -3, -8, \quad -4, -6.$

The only pair whose sum is -10 is $-4, -6$. Thus

$$x^2 - 10x + 24 = (x - 4)(x - 6).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x - 4)(x - 6) &= x^2 - 6x - 4x + 24\\ &= x^2 - 10x + 24.)\end{aligned}$$

(b) The quadratic is $t^2 - 4t + 3$.

The pair $-1, -3$ has product 3 and sum -4. Thus

$$t^2 - 4t + 3 = (t - 1)(t - 3).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(t - 1)(t - 3) &= t^2 - 3t - t + 3\\ &= t^2 - 4t + 3.)\end{aligned}$$

(c) The quadratic is $x^2 - 6x + 9$.

The negative factor pairs of 9 are

$-1, -9, \quad -3, -3.$

The only pair whose sum is -6 is $-3, -3$. Thus

$$x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2.$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x - 3)(x - 3) &= x^2 - 3x - 3x + 9\\ &= x^2 - 6x + 9.)\end{aligned}$$

Activity 19

(a) The quadratic is $x^2 - x - 2$.

The pair $1, -2$ has sum -1 and product -2. Thus

$$x^2 - x - 2 = (x + 1)(x - 2).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(x + 1)(x - 2) &= x^2 - 2x + x - 2\\ &= x^2 - x - 2.)\end{aligned}$$

(b) The quadratic is $u^2 + 4u - 12$.

The factor pairs of -12 are

$-1, 12, \quad -2, 6, \quad -3, 4,$
$1, -12, \quad 2, -6, \quad 3, -4.$

The only pair whose sum is 4 is $-2, 6$. Thus

$$u^2 + 4u - 12 = (u - 2)(u + 6).$$

(Check: Multiplying out the brackets gives

$$\begin{aligned}(u - 2)(u + 6) &= u^2 + 6u - 2u - 12\\ &= u^2 + 4u - 12.)\end{aligned}$$

Activity 20

(a) $x^2 - x = x(x - 1)$

(b) $u^2 - 16 = u^2 - 4^2 = (u - 4)(u + 4)$

(c) $t^2 - 9t = t(t - 9)$

(d) $x^2 + 10x + 25 = (x + 5)^2$

Activity 21

(a) The equation is: $x^2 + 3x + 2 = 0$

Factorise: $(x + 1)(x + 2) = 0$

So: $x + 1 = 0$ or $x + 2 = 0$

So: $x = -1$ or $x = -2$

(Check: When $x = -1$,

$$\begin{aligned}x^2 + 3x + 2 &= (-1)^2 + 3 \times (-1) + 2\\ &= 1 - 3 + 2 = 0.\end{aligned}$$

When $x = -2$,

$$\begin{aligned}x^2 + 3x + 2 &= (-2)^2 + 3 \times (-2) + 2\\ &= 4 - 6 + 2 = 0.)\end{aligned}$$

(b) The equation is: $x^2 - 10x + 24 = 0$

Factorise: $(x - 4)(x - 6) = 0$

So: $x - 4 = 0$ or $x - 6 = 0$

So: $x = 4$ or $x = 6$

(Check: When $x = 4$,

$$\begin{aligned}x^2 - 10x + 24 &= 4^2 - 10 \times 4 + 24\\ &= 16 - 40 + 24 = 0.\end{aligned}$$

When $x = 6$,

$$\begin{aligned}x^2 - 10x + 24 &= 6^2 - 10 \times 6 + 24\\ &= 36 - 60 + 24 = 0.)\end{aligned}$$

(c) The equation is: $t^2 - 16 = 0$

Factorise: $(t-4)(t+4) = 0$

So: $t - 4 = 0$ or $t + 4 = 0$

So: $t = 4$ or $t = -4$

(Check: When $t = 4$,

$t^2 - 16 = 4^2 - 16 = 16 - 16 = 0.$

When $t = -4$,

$t^2 - 16 = (-4)^2 - 16 = 16 - 16 = 0.$)

(Factorising is not the best way to solve this equation. It is quicker to rearrange it as $t^2 = 16$ and take square roots.)

(d) The equation is: $u^2 - u - 12 = 0$

Factorise: $(u-4)(u+3) = 0$

So: $u - 4 = 0$ or $u + 3 = 0$

So: $u = 4$ or $u = -3$

(Check: When $u = 4$,

$$\begin{aligned} u^2 - u - 12 &= 4^2 - 4 - 12 \\ &= 16 - 4 - 12 = 0. \end{aligned}$$

When $u = -3$,

$$\begin{aligned} u^2 - u - 12 &= (-3)^2 - (-3) - 12 \\ &= 9 + 3 - 12 = 0.) \end{aligned}$$

(e) The equation is: $x^2 - 6x + 9 = 0$

Factorise: $(x-3)(x-3) = 0$

So: $x - 3 = 0$

So: $x = 3$

(Check: When $x = 3$,

$$\begin{aligned} x^2 - 6x + 9 &= 3^2 - 6 \times 3 + 9 \\ &= 9 - 18 + 9 = 0.) \end{aligned}$$

(f) The equation is: $x^2 - 9x = 0$

Factorise: $x(x-9) = 0$

So: $x = 0$ or $x - 9 = 0$

So: $x = 0$ or $x = 9$

(Check: When $x = 0$,

$x^2 - 9x = 0^2 - 9 \times 0 = 0 - 0 = 0.$

When $x = 9$,

$x^2 - 9x = 9^2 - 9 \times 9 = 81 - 81 = 0.$)

Activity 22

(a) In this case 3 is a factor of each of the coefficients, so

$$\begin{aligned} 3x^2 - 3x - 36 &= 3(x^2 - x - 12) \\ &= 3(x-4)(x+3). \end{aligned}$$

(b) In this case -5 is a factor of each of the coefficients, so

$$\begin{aligned} -5x^2 + 15x - 10 &= -5(x^2 - 3x + 2) \\ &= -5(x-1)(x-2). \end{aligned}$$

Activity 23

(a) The quadratic expression is $2x^2 - 5x + 3$.

You can use the first method to look for a possible factorisation of the form

$$2x^2 - 5x + 3 = (2x \,\sqcup\,)(x \,\sqcup\,).$$

The integers in the gaps must multiply together to give 3, and the possible factor pairs of 3 are

$$1, 3, \quad -1, -3.$$

These two factor pairs lead to four possible cases:

$$(2x+1)(x+3) \quad \text{or} \quad (2x+3)(x+1),$$
$$(2x-1)(x-3) \quad \text{or} \quad (2x-3)(x-1).$$

By multiplying out each of these pairs of brackets in turn, we find that one of these cases gives the required factorisation, specifically,

$$(2x-3)(x-1) = 2x^2 - 5x + 3.$$

(b) The quadratic expression is $6x^2 + 7x - 3$.

You can use the first method to look for possible factorisations of the form

$$6x^2 + 7x - 3 = (6x \,\sqcup\,)(x \,\sqcup\,)$$

or

$$6x^2 + 7x - 3 = (3x \,\sqcup\,)(2x \,\sqcup\,).$$

The integers in the gaps must multiply together to give -3, and the possible factor pairs of -3 are

$$1, -3, \quad -1, 3.$$

These two factor pairs and the two possible factorisations lead to 8 possible cases:

$$(6x+1)(x-3) \quad \text{or} \quad (6x-3)(x+1),$$
$$(6x-1)(x+3) \quad \text{or} \quad (6x+3)(x-1),$$
$$(3x+1)(2x-3) \quad \text{or} \quad (3x-3)(2x+1),$$
$$(3x-1)(2x+3) \quad \text{or} \quad (3x+3)(2x-1).$$

By multiplying out each of these pairs of brackets in turn, we find that one of these cases gives the required factorisation, namely,

$$(3x-1)(2x+3) = 6x^2 + 7x - 3.$$

(c) The quadratic expression is $8x^2 - 10x - 3$.

You can use the second method to find a factorisation.

First find two numbers whose product is $ac = 8 \times (-3) = -24$ and whose sum is $b = -10$. The possible factor pairs of -24 (excluding those involving 24 or -24) are

$$-2, 12, \quad 2, -12, \quad -3, 8, \quad 3, -8, \quad -4, 6, \quad 4, -6.$$

Now choose a factor pair whose sum is -10. The only such pair is $2, -12$.

Next rewrite the quadratic expression, splitting the middle term $-10x$ into two terms using the factor pair $2, -12$, as follows:

$$8x^2 - 10x - 3 = 8x^2 + 2x - 12x - 3.$$

The required factorisation of $8x^2 - 10x - 3$ can now be obtained by taking out common factors:

$$\begin{aligned} 8x^2 - 10x - 3 &= \underbrace{8x^2 + 2x}\ \underbrace{-\,12x - 3} \\ &= 2x(4x+1) - 3(4x+1) \\ &= (2x-3)(4x+1). \end{aligned}$$

So

$$8x^2 - 10x - 3 = (2x-3)(4x+1).$$

Activity 24

(a) Since

$$2x^2 - 5x + 3 = (2x-3)(x-1),$$

the solutions of this equation satisfy

$$2x - 3 = 0 \quad \text{or} \quad x - 1 = 0,$$

so they are

$$x = \tfrac{3}{2} \quad \text{and} \quad x = 1.$$

(b) Since

$$6x^2 + 7x - 3 = (3x-1)(2x+3),$$

the solutions of this equation satisfy

$$3x - 1 = 0 \quad \text{or} \quad 2x + 3 = 0,$$

so they are

$$x = \tfrac{1}{3} \quad \text{and} \quad x = -\tfrac{3}{2}.$$

(c) Since

$$8x^2 - 10x - 3 = (2x-3)(4x+1),$$

the solutions of this equation satisfy

$$2x - 3 = 0 \quad \text{or} \quad 4x + 1 = 0,$$

so they are

$$x = \tfrac{3}{2} \quad \text{and} \quad x = -\tfrac{1}{4}.$$

Activity 25

(a) Since this is a right-angled triangle, the side lengths satisfy Pythagoras Theorem:

$$(x+8)^2 = x^2 + (x+7)^2.$$

(b) Expanding both brackets in the above equation gives

$$\begin{aligned} x^2 + 16x + 64 &= x^2 + x^2 + 14x + 49 \\ &= 2x^2 + 14x + 49. \end{aligned}$$

Subtracting $x^2 + 16x + 64$ from both sides gives $0 = x^2 - 2x - 15$, so x satisfies the quadratic equation

$$x^2 - 2x - 15 = 0.$$

Factorising gives

$$x^2 - 2x - 15 = (x-5)(x+3) = 0,$$

so the solutions of this quadratic equation satisfy

$$x - 5 = 0 \quad \text{or} \quad x + 3 = 0.$$

Hence

$$x = 5 \quad \text{or} \quad x = -3.$$

Since a negative number makes no sense as a solution, the answer must be 5 cm.

(Check: This gives sides of lengths 5 cm, 12 cm and 13 cm, and the numbers 5, 12 and 13 do satisfy

$$13^2 = 169 = 5^2 + 12^2,$$

as required.)

Activity 26

The width of the garden is x metres and its length is $2x$ metres. Since the path is 1 metre wide, the dimensions of the rectangular lawn in metres are $x - 2$ wide and $2x - 2$ long. Hence

$$(x-2)(2x-2) = 40.$$

Expanding the brackets in this equation gives

$$2x^2 - 6x + 4 = 40,$$

so x satisfies the quadratic equation

$$2x^2 - 6x - 36 = 0; \quad \text{that is,} \quad x^2 - 3x - 18 = 0.$$

Factorising gives

$$x^2 - 3x - 18 = (x-6)(x+3) = 0,$$

so the solutions of this quadratic equation are

$$x = 6 \quad \text{and} \quad x = -3.$$

Since a negative number makes no sense as a solution, the garden must have width 6 m.

(Check: This gives width 6 m and length 12 m for the garden, so width 4 m and length 10 m for the lawn, and 4 and 10 do satisfy $4 \times 10 = 40$, as required.)

Activity 27

(a) $$\frac{x^4}{x^9} = \frac{\overset{1}{\cancel{x^4}}}{\underset{x^5}{\cancel{x^9}}} = \frac{1}{x^5}$$

(b) $$\frac{20a^2b}{15ab^2} = \frac{\overset{4}{\cancel{20}}\overset{a}{\cancel{a^2}}\overset{1}{\cancel{b}}}{\underset{3}{\cancel{15}}\underset{1}{\cancel{a}}\underset{b}{\cancel{b^2}}} = \frac{4a}{3b}$$

(c) $$\frac{x^2 + 6x}{3x^2} = \frac{x(x+6)}{3x^2} = \frac{\overset{1}{\cancel{x}}(x+6)}{3\underset{x}{\cancel{x^2}}} = \frac{x+6}{3x}$$

(d) $$\begin{aligned} \frac{u^2 - 4}{u^2 + 4u + 4} &= \frac{(u-2)(u+2)}{(u+2)^2} \\ &= \frac{(u-2)\overset{1}{\cancel{(u+2)}}}{\underset{u+2}{\cancel{(u+2)^2}}} = \frac{u-2}{u+2} \end{aligned}$$

Activity 28

(a) $$\frac{6}{y}+\frac{1}{y}=\frac{6+1}{y}=\frac{7}{y}$$

(b) $$\frac{x}{a^2}+\frac{y}{a^2}-\frac{z}{a^2}=\frac{x+y-z}{a^2}$$

(c) Use $6xy$ as a common denominator:
$$\frac{a}{2x}-\frac{b}{3y}=\frac{3ay}{6xy}-\frac{2bx}{6xy}=\frac{3ay-2bx}{6xy}.$$

(d) Use $3x(x+3)$ as a common denominator:
$$\begin{aligned}\frac{1}{3x}-\frac{2}{x+3}&=\frac{x+3}{3x(x+3)}-\frac{6x}{3x(x+3)}\\&=\frac{x+3-6x}{3x(x+3)}\\&=\frac{-5x+3}{3x(x+3)}.\end{aligned}$$

Activity 29

(a) Use x^4 as a common denominator:
$$\frac{5}{x^4}-\frac{4}{x^2}=\frac{5}{x^4}-\frac{4x^2}{x^4}=\frac{5-4x^2}{x^4}.$$

(b) Use $x(x+1)$ as a common denominator:
$$\begin{aligned}\frac{1}{x^2+x}-\frac{1}{x+1}&=\frac{1}{x(x+1)}-\frac{x}{x(x+1)}\\&=\frac{1-x}{x(x+1)}.\end{aligned}$$

(c) Use $b+3$ as a common denominator:
$$\begin{aligned}a+\frac{a}{b+3}&=\frac{a}{1}+\frac{a}{b+3}\\&=\frac{a(b+3)}{b+3}+\frac{a}{b+3}\\&=\frac{a(b+3)+a}{b+3}=\frac{ab+4a}{b+3}=\frac{a(b+4)}{b+3}.\end{aligned}$$

Activity 30

(a) $$\frac{p^2}{q^2}\times\frac{p}{q}=\frac{p^2\times p}{q^2\times q}=\frac{p^3}{q^3}$$

(b) $$\frac{p^2}{q^2}\div\frac{p}{q}=\frac{p^2}{q^2}\times\frac{q}{p}=\frac{p^2\times q}{q^2\times p}=\frac{p^2q}{q^2p}=\frac{p}{q}$$

(c) $$\frac{9ax^2}{b}\times\frac{b^3}{4xy^2}=\frac{9ax^2\times b^3}{b\times 4xy^2}=\frac{9axb^2}{4y^2}$$

(d) $$\begin{aligned}\frac{9ax^2}{b}\div\frac{b^3}{4xy^2}&=\frac{9ax^2}{b}\times\frac{4xy^2}{b^3}\\&=\frac{9ax^2\times 4xy^2}{b\times b^3}=\frac{36ax^3y^2}{b^4}\end{aligned}$$

(e) $$\begin{aligned}8u^2\times\frac{2}{u^2+u}&=\frac{8u^2}{1}\times\frac{2}{u^2+u}\\&=\frac{16u^2}{u^2+u}=\frac{16u^2}{u(u+1)}=\frac{16u}{u+1}\end{aligned}$$

Activity 31

(a) By the rule for dividing by a fraction,
$$\frac{F}{f}=\frac{GM}{R^2}\div\frac{Gm}{r^2}=\frac{GM}{R^2}\times\frac{r^2}{Gm}=\frac{Mr^2}{mR^2}.$$

(b) We substitute $M=80m$ and $R=4r$ in the formula obtained in part (a):
$$\frac{F}{f}=\frac{80mr^2}{m(4r)^2}=\frac{80mr^2}{16mr^2}=5,$$
so $F=5f$.

Activity 32

(a) The equation is
$$\frac{4}{x-3}=\frac{12}{x+1}.$$
First multiply by $(x-3)(x+1)$ (assuming that $x\neq 3$ and $x\neq -1$):
$$(x-3)(x+1)\frac{4}{x-3}=(x-3)(x+1)\frac{12}{x+1}.$$
Then cancel and multiply out the brackets:
$$4(x+1)=12(x-3);$$
that is,
$$4x+4=12x-36.$$
Finally, rearrange the equation as
$$8x=40.$$
This gives $x=5$ (which satisfies the assumptions).

(b) The equation is
$$\frac{t}{5}=\frac{2}{t+3}.$$
First multiply by $5(t+3)$ (assuming that $t\neq -3$):
$$5(t+3)\frac{t}{5}=5(t+3)\frac{2}{t+3}.$$
Then cancel and multiply out the brackets:
$$t(t+3)=10;$$
that is,
$$t^2+3t=10.$$
Rearrange:
$$t^2+3t-10=0.$$
Factorise:
$$(t+5)(t-2)=0.$$
This gives $t=-5$ or $t=2$ (which satisfy the assumptions).

(c) The equation is
$$\frac{4}{x}+2x=9.$$
First multiply by x (assuming that $x\neq 0$):
$$x\left(\frac{4}{x}+2x\right)=9x.$$

Then multiply out the brackets:

$$4 + 2x^2 = 9x.$$

Rearrange:

$$2x^2 - 9x + 4 = 0.$$

Factorise:

$$(2x - 1)(x - 4) = 0.$$

This gives $x = \frac{1}{2}$ or $x = 4$ (which satisfy the assumptions).

(d) The equation is

$$\frac{10}{x} + \frac{20}{x + 40} = \frac{1}{2}.$$

To clear all the fractions at once, multiply all terms by the product $2x(x + 40)$ of their denominators (assuming that $x \neq 0$ and $x \neq -40$):

$$2x(x + 40)\frac{10}{x} + 2x(x + 40)\frac{20}{x + 40} = 2x(x + 40)\frac{1}{2}.$$

Cancel:

$$20(x + 40) + 40x = x(x + 40).$$

Expand the brackets:

$$20x + 800 + 40x = x^2 + 40x.$$

Rearrange:

$$x^2 - 20x - 800 = 0.$$

Factorise:

$$(x - 40)(x + 20) = 0.$$

This gives $x = 40$ or $x = -20$ (which satisfy the assumptions).

Activity 33

(a) The equation is

$$v = \frac{2}{u + 3} + 4.$$

Multiply both sides by $u + 3$ (assuming that $u + 3 \neq 0$):

$$(u + 3)v = (u + 3)\left(\frac{2}{u + 3} + 4\right).$$

Multiply out brackets and cancel:

$$\begin{aligned} uv + 3v &= 2 + 4(u + 3) \\ &= 4u + 14. \end{aligned}$$

Rearrange:

$$uv - 4u = 14 - 3v.$$

Factorise:

$$u(v - 4) = 14 - 3v.$$

Divide by $v - 4$ (assuming that $v - 4 \neq 0$):

$$u = \frac{14 - 3v}{v - 4}.$$

(b) The equation is

$$y = \frac{20}{3x} - \frac{3}{2}.$$

Multiply both sides by $6x$ (assuming that $x \neq 0$):

$$6xy = 6x\left(\frac{20}{3x} - \frac{3}{2}\right).$$

Multiply out the brackets and cancel:

$$6xy = 40 - 9x.$$

Rearrange:

$$6xy + 9x = 40.$$

Factorise:

$$x(6y + 9) = 40.$$

Divide by $6y + 9$ (assuming that $6y + 9 \neq 0$):

$$x = \frac{40}{6y + 9}.$$

Activity 34

(a) **(i)** The formula is

$$V = \tfrac{1}{3}\pi r^2 h.$$

Multiply by 3:	$3V = \pi r^2 h$
Divide by πh: (since $h \neq 0$)	$\dfrac{3V}{\pi h} = r^2$
Swap the sides:	$r^2 = \dfrac{3V}{\pi h}$
Raise to the power $\frac{1}{2}$:	$r = \left(\dfrac{3V}{\pi h}\right)^{\frac{1}{2}}$

(The last step is valid since the right-hand side is positive.)

(ii) If $h = 1\,\text{m}$ and $V = 2\,\text{m}^3$, then

$$\begin{aligned} r &= \left(\frac{3 \times 2}{\pi \times 1}\right)^{\frac{1}{2}} \\ &= 1.38\,\text{m (to 3 s.f.)} \end{aligned}$$

(b) The formula is

$$T^2 = \frac{MS^3}{cD}.$$

Multiply by cD (since $cD \neq 0$):	$cDT^2 = MS^3$
Divide by M (since $M \neq 0$):	$\dfrac{cDT^2}{M} = S^3$
Swap the sides:	$S^3 = \dfrac{cDT^2}{M}$
Raise to the power $\frac{1}{3}$:	$S = \left(\dfrac{cDT^2}{M}\right)^{\frac{1}{3}}$

(The last step is valid since the right-hand side is positive.)